CONTRACTING WITH HETEROGENEOUS EXTERNALITIES

WEB APPENDIX

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Proposition 8 *Consider a participation problem* $(N, \mathbf{w}, \mathbf{c})$. *Let* $(N, \mathbf{w}^+, \mathbf{c})$ *be a participation problem such that* $w_i^+(j) = w_i(j)$ *if* $w_i(j) \geq 0$ *and* $w_i^+(j) = 0$ *if* $w_i(j) < 0$, *and let* u^+ *be the optimal full implementation contracts of* $(N, \mathbf{w}^+, \mathbf{c})$. *Let* $(N, \mathbf{w}^-, \mathbf{0})$ *be a participation problem such that* $w_i^-(j) = w_i(j)$ *if* $w_i(j) < 0$ *and* $w_i^-(j) = 0$ *if* $w_i(j) \geq 0$, *and let* $\mathbf{u}^-$ *be the optimal full implementation contracts of* $(N, \mathbf{w}^-, \mathbf{0})$. *Then, the decomposition contracting scheme* $\mathbf{v} = \mathbf{u}^+ + \mathbf{u}^-$ *induces a unique full participation equilibrium. Moreover, if agents satisfy symmetry and transitivity with respect to the non-averse relation,* $\mathbf{v}$ *is the optimal contracting scheme.*

Proof of Proposition 8 To prove the proposition we use the following lemma.
Lemma 8.1: Consider a participation problem $(N, \mathbf{w}, \mathbf{c})$ with mixed externalities. Then the decomposition contracting scheme sustains full participation as a unique equilibrium.
Proof: Consider a decomposition contracting scheme $\mathbf{v}$ of the participation problem $(N, \mathbf{w}, \mathbf{c})$, when $\mathbf{w}$ includes mixed externalities. By definition $v_i = u_i^+ + u_i^-$, where $\mathbf{u}^+$ is the optimal full implementation contracting scheme for the positive participation problem, and $\mathbf{u}^-$ is the optimal contracting scheme for the negative participation problem. Let $\phi = \{i_1, ..., i_N\}$ be the optimal ranking in the positive participation problem $(N, \mathbf{w}^+, \mathbf{c})$. As demonstrated in Proposition 6, the optimal contracting scheme for the negative participation problem is full compensation of agents for the negative externalities, i.e., $u_i^- = \sum_{j \in D_i} |w_i(j)|$ where $D_i = \{j \mid w_i(j) < 0 \; s.t. \; j \in N\}$. We will demonstrate that $\mathbf{v}$ sustains full participation as a unique equilibrium. To deter no-participation equilibrium, at least a single agent has to participate, regardless of the participation choices of the other agents. Agent i_1 gets $u_{i_1}^+ = c$, and $u_{i_1}^- = \sum_{j \in D_{i_1}} \left| w_{i_1}(j) \right|$; therefore, regardless of the others' choices, agent i_1 will participate in the initiative. Next, it is necessary to prevent an equilibrium in which only a single agent participates, and note that given agent i_1's participation, agent i_2 will choose to participate, regardless of the choices of agents $i_3, ..., i_N$. This is true since $u_{i_2}^+ = c - w_{i_2}^+(i_1)$ and $u_{i_1}^- = \sum_{j \in D_{i_2}} \left| w_{i_2}(j) \right|$, and, given the participation of agent i_1 and the compensation for negative externalities, agent i_2 will participate. Applying the same logic for all subsequent agents, we get that the only possible equilibrium is full participation, since all agents are willing to participate given the participation of the agents who preceded them in the optimal ranking of the positive participation problem, and since they are being compensated for negative externalities. **End of Lemma.**

Now we need to show that if the matrix of externalities satisfies symmetry and transitivity with respect to the *non-averse* relation, then the decomposition contracting scheme

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generates the optimal set of contracts for full participation. Note that since we have defined $\omega_{ii} = 0$ for each agent i, the *non-averse* relation $\succeq$ is reflexive (i.e., $i \succeq i$) and together with the symmetry and transitivity assumptions it is an equivalence relation. Therefore the set of agents can be partitioned into equivalence classes $S_1, ..., S_m$ such that $\cup_j S_j = N$ and $S_j \cap S_i = \emptyset$ for all i, j, and for each $k, l \in S_i$ we have that k and l induce non-negative externalities on each other. Let us now define m new participation problems by restricting the matrix of externalities to the set S_j where $1 \leq j \leq m$. All these problems are non-negative and their optimal contracting schemes are all DAC. Let u_j^+ be an optimal contracting scheme for the problem defined on the set S_j, which is a vector of size $\#S_j$. Write $\mathbf{u}^+ = (u_1^+, ..., u_m^+)$. For each agent $i \in N$ denote by d_i the total of negative externalities imposed on agent i, i.e., $d_i = \Sigma_{\{j;\omega_{ij}<0\}}\omega_{ij}$, and set $\mathbf{u}^- = (d_1, ..., d_n)$. We will show that $\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-$ is an optimal contracting scheme for the mixed problem.

Let $\mathbf{w}$ be the original matrix of externalities and let $\mathbf{w} = \mathbf{w}^+ + \mathbf{w}^-$ be the decomposition of $\mathbf{w}$, where $\mathbf{w}^+$ is a non-negative matrix and $\mathbf{w}^-$ is a non-positive matrix. Clearly, $\mathbf{u}^+$ is an optimal contracting scheme for $\mathbf{w}^+$. This is true since under $\mathbf{w}^+$ the externalities between agents of the same class are non-negative and between agents of different classes they are zero. By a similar argument, $\mathbf{u}^-$ is the optimal contracting scheme for $\mathbf{w}^-$. Hence $\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-$ is a decomposition contracting scheme and as such by Lemma 1 it sustains full participation as a unique equilibrium.

To show that $\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-$ is optimal for the participation problem $(N, \mathbf{w}, \mathbf{c})$ we have to show that the principal cannot extract more from the agents in the full participation unique equilibrium. Let ij denote the agent at the $i-$th location in the optimal order of the DAC contracting scheme within class S_j. Consider first an alternative contracting scheme $\mathbf{u}'$, which is identical to $\mathbf{u}$ in terms of incentives for all agents except for a single agent, who is placed first in the optimal order of some class j. This agent is getting less in $\mathbf{u}'$, i.e., $u'_{1j} < u_{1j}$. Under the contracting scheme $\mathbf{u}'$ there exists a Nash equilibrium in which the set of participating agents is $N \backslash S_j$. This is true since $1j$ does not have a dominant strategy to participate and therefore all agents in S_j fail to participate in a Nash equilibrium in the game restricted to S_j, and hence also in the entire game. Furthermore, any agent in $N \backslash S_j$ chooses to participate when S_j participates, and even more so if the S_j class stays out (given the negative externalities between S_j and the rest). Hence $\mathbf{u}'$ does not qualify as an optimal contracting scheme. A similar argument holds whenever we reduce the payoff of more than one agent while limiting ourselves to one class only, say S_j. As we proved in Propositions 2 and 4, this induces an equilibrium on the game restricted to S_j with partial participation and this equilibrium also applies to the entire game; hence $\mathbf{u}'$ does not induce a unique full participation equilibrium.

Now consider the case where the alternative contracting scheme $\mathbf{u}'$ reduces the payoff to more than a single agent and from more than a single class. Assume first that for some k classes $\{l_1, ..., l_k\}$, with $k \leq m$, agents $1j$, $j \in \{l_1, ..., l_k\}$ are paid less in $\mathbf{u}'$ than in $\mathbf{u}$ and that all other agents are paid the same. It must be the case that under $\mathbf{u}'$ full participation is a Nash equilibrium; otherwise it is definitely not the optimal unique full participation Nash equilibrium. We assume therefore that under $\mathbf{u}'$ full participation is a Nash equilibrium and we shall show that it is nevertheless not a unique equilibrium. Specifically, we will show that under $\mathbf{u}'$ we can construct a Nash equilibrium with the set of participants being $N \backslash S_{j^*}$, when S_{j*} is an arbitrary class such that $j^* \in \{l_1, ..., l_k\}$.

We first note that there exists an equilibrium in which none of the members in S_{j*} participate. Indeed, given that all agents in $N \backslash S_{j*}$ participate, participation is no longer a dominant strategy for agent $1j^*$ and hence there exists an equilibrium in which all agents in S_{j*} are staying out of the game restricted to S_{j*}, and therefore this equilibrium holds also for the entire game. Since for all agents in $N \backslash S_{j*}$ it was a best response to participate even when all members in S_{j*} (with whom they have negative externalities) participate, it is certainly still a best response for these agents when S_{j*} do not participate. Hence partial participation is a Nash equilibrium and we obtained the desired contradiction. Next, we assume a contracting scheme $\mathbf{u}'$ in which an arbitrary group of agents is paid less than the payoff in $\mathbf{u}$. Let j^* be a class such that $j^* \in \{l_1, ..., l_k\}$ in which some agent ij is paid less. Using the argument in the first part of the proof and in Propositions 2 and 4, there is a Nash equilibrium in the game restricted to S_{j*} where there is only partial participation. Furthermore, the profile in which the set of participants is exactly $N \backslash S_{j*}$ in addition to the subset of participants in S_{j*} is a Nash equilibrium in the entire game. This again follows from the fact that under $\mathbf{u}'$ the best response of each agent in $N \backslash S_{j*}$ to full participation by the rest must be to participate, and $N \backslash S_{j*}$ will definitely participate if a subset of S_{j*} does not participate.

We have shown so far that for a given contracting scheme $\mathbf{u}'$ in which some agents get less than $\mathbf{u}$ and the rest get the same as $\mathbf{u}$, there is an equilibrium in which some agents do not participate. It is therefore left to show that for a given contracting scheme $\mathbf{u}'$ in which we reduce the payoff for some agents and increase the payoff for others either we still have an equilibrium in which some agents do not participate or alternatively the total payment of the principal increases. Consider first that $\mathbf{u}$ is unique; hence the optimal solution u_+^j for each positive participation problem in each class is unique. Assume that $\mathbf{u}'$ is achieved by lowering the payoff for the agent ranked first in class j^*, i.e., agent $1j^*$. To rule out the equilibrium in which class S_{j^*} is not participating it is necessary to increase the payoff of another agent, say kj^*, in such a way that he will have a dominant strategy to participate, and induce the participation of the agent whose payoff was lowered. This means moving the kj^* agent up to become first in the class. But this must cost more to the principal as $\mathbf{v}^{j*}$ is unique, a contradiction. Alternatively, if $\mathbf{u}$ is not unique, then there are multiple solutions for each positive participation problem. The argument is very similar. Say that $\mathbf{u}$ was chosen from a group of decomposition contracting schemes. Again, to avoid partial participation equilibrium it is necessary to provide higher incentives to agent kj^*, who becomes first. If the total payment is identical to the payment in $\mathbf{v}^{j*}$ then we have reached a different optimal solution to the participation problem of class j^*. However, if we do not reach a different decomposition contracting scheme, this implies that the payment in this case is higher. The same logic can be applied to situations in which we lower the payoffs for other agents within the different classes. Therefore we conclude that the decomposition contracting scheme is the optimal contracting scheme that sustains full participation as a unique equilibrium.