Online Appendix for "Contracting in Vague Environments"

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Including the case of $\alpha_P \in \{0,1\}$ in the main result

This online Appendix states the result in Theorem 2 of the main paper for $(\alpha_A, \alpha_P) \in [0,1] \times [0,1]$, and includes the proof of the Theorem when $\alpha_P \in \{0,1\}$. The proof for $\alpha_P \in (0,1)$ is in the main paper.

Theorem 2' (Optimal contracts)**.** *The optimal contract varies across (α_A, α_P)-space as depicted in Figure 1 in the main paper. Specifically:*

If $\alpha_P \leq \alpha_A \leq 1-\alpha_P$, the optimal contract is a "sell the firm to the agent" contract with $e_L^* = e_L^o$ and $e_H^* = e_H^o$.

If $\alpha_A < \alpha_P$ and $\alpha_A \leq 1-\alpha_P$, the optimal contract has distortion at the bottom with $e_H^* = e_H^o$ and $e_L^* < e_L^o$.

If $\alpha_A \geq \alpha_P$ and $\alpha_A > 1-\alpha_P$, the optimal contract has distortion at the top with $e_L^* = e_L^o$ and $e_H^* > e_H^o$.

If $1-\alpha_P < \alpha_A < \alpha_P$, there exists $\hat{\alpha} \in (1-\alpha_P, \alpha_P)$ such that the optimal contract has distortion at the bottom for all $\alpha_A < \hat{\alpha}$ and has distortion at the top for all $\alpha_A \geq \hat{\alpha}$.[1]

[1] There is an exception: if $a = 0$, $\alpha_P = 1$, and $\alpha_A > 0$, there is no solution to the principal's problem. The principal would want to offer a contract that has distortion at the top. With that contract, she would assign zero weight to the agent being of type x_H. She would therefore want to distort effort for the high-efficiency type towards infinity, combined with paying this type a correspondingly higher wage and require a correspondingly higher price for the firm from the low-efficiency type. Since $\alpha_A > 0$, the agent assigns positive weight to being of high-efficiency type, thus the contract, with sufficiently high wage to type x_H, would satisfy his participation constraint. Of course, this problem arises only because I have assumed no upper bound on the agent's effort.

Proof of Theorem 2' when $\alpha_P = 0$ or $\alpha_P = 1$

The arguments given above the Lagrangian in Appendix B apply here as well. Equation numbers refer to equations either in the main paper or in this online Appendix.

Proof when $\alpha_P = 0$

The first-order conditions for the principal's problem are given by (B1) through (IC_L) with $\alpha_P \overline{p}_{H,P}(h) + (1-\alpha_P)\underline{p}_{H,P}(h) = \underline{p}_{H,P}(h)$, where (B3), (B4), (PC), (IC_H), and (IC_L) hold with equality if, respectively, e_H^*, e_L^*, γ^*, λ_H^*, and λ_L^* are strictly greater than zero. It follows from (B1) and (B2) that $\gamma^* = 1$. The analysis of the first-order conditions can again be broken down into 4 cases, which are to be compared to the corner contracts.

Case 1: $\lambda_{\mathbf{L}} > \mathbf{0}$ and $\lambda_{\mathbf{H}} = \mathbf{0}$. In this case, (B2) implies that $\underline{p}_{H,P}(h) < 1$ and (B4) implies that $e_L^* > 0$. Conditions (B2) and (B4) then together imply that $e_L^* = e_L^o$. Also, (IC_L) implies that $w_L^* - g(e_L^o, x_L) = w_H^* - g(e_H^*, x_L)$.

Suppose first that $\underline{p}_{H,P}(h) > 0$. Then (B3) implies that $e_H^* > 0$ and (B3) together with (B1) imply that $e_H^* > e_H^o$. We hence have distortion at the top. Condition (B2) implies that

$$\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h) > \alpha_P \overline{p}_{H,P}(h) + (1-\alpha_P)\underline{p}_{H,P}(h). \tag{C1}$$

The principal will be best off if the agent is of type x_L, by the same argumentation as used below expressions (B8) and (B9) in Appendix B. Thus, $\underline{p}_{H,P}(h) = b$. Then (C1) is equivalent to $\alpha_A b + (1-\alpha_A)a > b$, which is a contradiction since $\alpha_A \in [0,1]$ and $b > a$. It can therefore be ruled out that $\underline{p}_{H,P}(h) > 0$.

Suppose instead that $\underline{p}_{H,P}(h) = 0$. Then (B1) and (B3) imply that $g_e(e_H^*, x_H) \geq g_e(e_H^*, x_L)$, which only holds if $e_H^* = 0$. Using this, (IC_H) becomes $w_H^* \geq w_L^* - g(e_L^o, x_H)$ and (IC_L) becomes $w_L^* - g(e_L^o, x_L) = w_H^*$, which together imply that $e_L^o = 0$. This contradicts $e_L^o > 0$.

Since both $\underline{p}_{H,P}(h) > 0$ and $\underline{p}_{H,P}(h) = 0$ lead to contradictions, the conclusion is that Case 1 will never be prevailing when $\alpha_P = 0$.

Case 2: $\lambda_{\mathbf{H}} > \mathbf{0}$ and $\lambda_{\mathbf{L}} = \mathbf{0}$. In this case, (B1) implies that $\underline{p}_{H,P}(h) > 0$ and (B3) implies that $e_H^* > 0$. Conditions (B1) and (B3) then together imply that $e_H^* = e_H^o$. Also, (IC_H) implies that $w_H^* - g(e_H^o, x_H) = w_L^* - g(e_L^*, x_H)$.

Suppose first that $\underline{p}_{H,P}(h) < 1$. Then (B4) implies that $e_L^* > 0$ and (B4) together with

(B2) imply that $e_L^* < e_L^o$. We hence have distortion at the bottom. Condition (B1) implies that

$$\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h) < \alpha_P \overline{p}_{H,P}(h) + (1-\alpha_P)\underline{p}_{H,P}(h). \tag{C2}$$

The principal will be best off if the agent is of type x_H, by the same argumentation as used below expressions (B14) and (B15) in Appendix B. Thus, $\underline{p}_{H,P}(h) = a$. Then (C2) is equivalent to $\alpha_A b + (1-\alpha_A)a < a$, which is a contradiction since $\alpha_A \in [0,1]$ and $b > a$. It can therefore be ruled out that $\underline{p}_{H,P}(h) < 1$.

Suppose instead that $\underline{p}_{H,P}(h) = 1$. Then (B2) and (B4) imply that

$$g_e(e_L^*, x_H) \le g_e(e_L^*, x_L). \tag{C3}$$

If $e_L^* > 0$ then (C3) implies that $g_e(e_L^*, x_H) \le g_e(e_L^*, x_L)$, which is only satisfied if $e_L^* = 0$. This is a contradiction. If instead $e_L^* = 0$, (IC_H) becomes $w_H^* - g(e_H^o, x_H) = w_L^*$ and (PC) becomes $(\alpha_A b + (1-\alpha_A)a)\left(w_H^* - g(e_H^o, x_H)\right) + (1 - (\alpha_A b + (1-\alpha_A)a))\, w_L^* = 0$, which together imply that $w_H^* = g(e_H^o, x_H)$. This in turn gives that $w_L^* = 0$. But then the principal's Bernoulli utility is $\pi(e_H^o) - g(e_H^o, x_H)$ when the agent is of type x_H and is zero when the agent is of type x_L, which means that the principal is best off when the agent is of type x_H. This contradicts that $\underline{p}_{H,P}(h) = 1$.

Since both $\underline{p}_{H,P}(h) < 1$ and $\underline{p}_{H,P}(h) = 1$ lead to contradictions, the conclusion is that Case 2 will never be prevailing when $\alpha_P = 0$.

Case 3: $\lambda_{\mathbf{H}} = \mathbf{0}$ and $\lambda_{\mathbf{L}} = \mathbf{0}$. In this case, (B1) implies that $\underline{p}_{H,P}(h) = \alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h)$. Condition (B3) implies that

$$\underline{p}_{H,P}(h)\pi_e(e_H^*) \le \left(\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h)\right) g_e(e_H^*, x_H)$$

and condition (B4) implies that

$$(1 - \underline{p}_{H,P}(h))\pi_e(e_L^*) \le \left(1 - \left(\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h)\right)\right) g_e(e_L^*, x_L).$$

Suppose first that $\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h) = 0$. This can only hold if $\alpha_A = 0$ and $a = 0$. We then have that $\underline{p}_{H,P}(h) = 0$, which implies that the principal must be best off when the agent is of type x_H. Condition (B4) then implies that $e_L^* = e_L^o$, while (PC) implies that $w_L^* = g(e_L^o, x_L)$. Furthermore, (IC_H) gives that $w_H^* - g(e_H^*, x_H) \ge w_L^* - g(e_L^o, x_H) > 0$, while (IC_L) gives that $w_H^* - g(e_H^*, x_L) \le 0$. These two conditions together imply that $e_H^* > 0$. Because $\alpha_P = 0$ and the principal is best off when the agent is of type x_H, the principal's utility is $OWEU_P = \pi(e_L^o) - g(e_L^o, x_L)$.

Suppose instead that $\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h) = 1$. This can only hold if $\alpha_A = 1$ and $b = 1$. We then have that $\underline{p}_{H,P}(h) = 1$, which implies that the principal must be best off when the agent is of type x_L. Condition (B3) then implies that $e_H^* = e_H^o$, while (PC) implies that $w_H^* = g(e_H^o, x_H)$. Furthermore, (IC_H) gives that $0 \geq w_L^* - g(e_L^o, x_H)$, while (IC_L) gives that $w_L^* - g(e_L^*, x_L) \geq w_H^* - g(e_H^o, x_L)$. Because $\alpha_P = 0$ and the principal is best off when the agent is of type x_L, the principal's utility is $OWEU_P = \pi(e_H^o) - g(e_H^o, x_H)$.

Finally, if $\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h) \in (0,1)$ we have a corner contract. These are considered in Appendix B.

Case 4: $\lambda_H > 0$ and $\lambda_L > 0$. In this case, (IC_H) and (IC_L) imply that $e_H^* = e_L^*$. Conditions (B1), (B2), (B3), and (B4) then imply that $e_H^* = e_L^* = 0$. But (B3) and (B4) imply that $e_H^* > 0$ or $e_L^* > 0$. Hence, Case 4 leads to a contradiction.

Corner contracts: $(\mathbf{w_H}, \mathbf{e_H}, \mathbf{w_L}, \mathbf{e_L})$ for which $\pi(\mathbf{e_H}) - \mathbf{w_H} = \pi(\mathbf{e_L}) - \mathbf{w_L}$. The analysis of the corner contracts is as in Appendix B, and the principal's utility with a corner contract is given by (B19).

I now have to establish which type of contract is optimal. Cases 1, 2, and 4 are ruled out. Comparing the principal's utility in Case 3 with her utility under a corner contract, the conclusion is that a corner contract is always optimal when $\alpha_P = 0$.

Proof when $\alpha_P = 1$

The first-order conditions for the principal's problem are again given by (B1) through (IC_L), this time with $\alpha_P \overline{p}_{H,P}(h) + (1-\alpha_P)\underline{p}_{H,P}(h) = \overline{p}_{H,P}(h)$, where (B3), (B4), (PC), (IC_H), and (IC_L) hold with equality if, respectively, e_H^*, e_L^*, γ^*, λ_H^*, and λ_L^* are strictly greater than zero. It follows from (B1) and (B2) that $\gamma^* = 1$. The analysis of the first-order conditions can be broken down into the same 4 cases as above, which are to be compared with the corner contracts.

Case 1: $\lambda_L > 0$ and $\lambda_H = 0$. In this case, (B2) implies that $\overline{p}_{H,P}(h) < 1$ and (B4) implies that $e_L^* > 0$. Conditions (B2) and (B4) then together imply that $e_L^* = e_L^o$. Also, (IC_L) implies that $w_L^* - g(e_L^o, x_L) = w_H^* - g(e_H^*, x_L)$. Condition (B2) also implies that

$$\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h) > \overline{p}_{H,P}(h). \tag{C4}$$

Given incentive compatibility, this is equivalent to $\alpha_A b + (1-\alpha_A)a > \overline{p}_{H,P}(h)$, which necessitates that $\alpha_A > 0$ and $\overline{p}_{H,P}(h) = a$. The principal will therefore be best off when the agent is of type x_L.

Suppose first that $\overline{p}_{H,P}(h) = 0$. Then (B3) together with (B1) imply that $g_e(e_H^*, x_L) \leq g_e(e_H^o, x_H)$, which is only satisfied if $e_H^* = 0$. Then (IC_H) and (IC_L) together imply that $e_L^o = 0$, which is a contradiction. Therefore, it must be that $\overline{p}_{H,P}(h) > 0$, that is, $a > 0$.

When $\overline{p}_{H,P}(h) > 0$, condition (B3) implies that $e_H^* > 0$, and (B3) together with (B1) imply that $e_H^* > e_H^o$. We hence have distortion at the top. Conditions (PC) and (IC_L) imply that

$$w_H^* = \big(\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h)\big) g(e_H^*, x_H) + \Big(1 - \big(\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h)\big)\Big) g(e_H^*, x_L)$$

and

$$w_L^* = g(e_L^o, x_L) + \big(\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h)\big)\left(g(e_H^*, x_H) - g(e_H^*, x_L)\right).$$

The principal's utility is given by (B10).

Case 2: $\boldsymbol{\lambda_H > 0}$ and $\boldsymbol{\lambda_L = 0}$. In this case, (B1) implies that $\overline{p}_{H,P}(h) > 0$ and (B3) implies that $e_H^* > 0$. Conditions (B1) and (B3) then together imply that $e_H^* = e_H^o$. Also, (IC_H) implies that $w_H^* - g(e_H^o, x_H) = w_L^* - g(e_L^*, x_H)$. Condition (B1) also implies that

$$\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h) < \overline{p}_{H,P}(h). \tag{C5}$$

Given incentive compatibility, this is equivalent to $\alpha_A b + (1-\alpha_A)a < \overline{p}_{H,P}(h)$, which necessitates that $\alpha_A < 1$ and $\overline{p}_{H,P}(h) = b$. The principal is will therefore be best off when the agent is of type x_H.

Suppose first that $\overline{p}_{H,P}(h) < 1$. Then (B4) implies that $e_L^* > 0$ and (B4) together with (B2) imply that $e_L^* < e_L^o$. The contract hence has distortion at the bottom. Conditions (PC) and (IC_H) imply that

$$w_L^* = \big(\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h)\big) g(e_L^*, x_H) + \Big(1 - \big(\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h)\big)\Big) g(e_L^*, x_L)$$

and

$$w_H^* = g(e_H^o, x_H) + \Big(1 - \big(\alpha_A \overline{p}_{H,A}(h) + (1-\alpha_A)\underline{p}_{H,A}(h)\big)\Big)\left(g(e_L^*, x_L) - g(e_L^*, x_H)\right).$$

The principal's utility is given by (B16).

Suppose instead that $\overline{p}_{H,P}(h) = 1$. Then (B4) together with (B2) imply that $g_e(e_L^*, x_H) \leq g_e(e_L^*, x_L)$. If $e_L^* > 0$, this implies that $e_L^* = 0$, a contradiction. Hence, it must be that $e_L^* = 0$. Then (PC) and (IC_H) imply that $w_L^* = 0$ and $w_H^* = g(e_H^o, x_H)$. The principal's utility is $\pi(e_H^o) - g(e_H^o, x_H)$. Note that $\overline{p}_{H,P}(h) = 1$ only when $b = 1$.

Case 3: $\lambda_\mathbf{H} = \mathbf{0}$ and $\lambda_\mathbf{L} = \mathbf{0}$. In this case, (B1) implies that $\overline{p}_{H,P}(h) = \alpha_A \overline{p}_{H,A}(h) + (1 - \alpha_A)\underline{p}_{H,A}(h)$. Condition (B3) implies that

$$\overline{p}_{H,P}(h)\pi_e(e_H^*) \leq \left(\alpha_A \overline{p}_{H,A}(h) + (1 - \alpha_A)\underline{p}_{H,A}(h)\right) g_e(e_H^*, x_H)$$

and condition (B4) implies that

$$(1 - \overline{p}_{H,P}(h))\pi_e(e_L^*) \leq \left(1 - \left(\alpha_A \overline{p}_{H,A}(h) + (1 - \alpha_A)\underline{p}_{H,A}(h)\right)\right) g_e(e_L^*, x_L).$$

Suppose first that $\alpha_A \overline{p}_{H,A}(h) + (1 - \alpha_A)\underline{p}_{H,A}(h) = 0$. This can only hold if $\alpha_A = 0$ and $a = 0$. We then have that $\overline{p}_{H,P}(h) = 0$, and thus principal must be best off when the agent is of type x_L. Condition (B4) then implies that $e_L^* = e_L^o$, while (PC) implies that $w_L^* = g(e_L^o, x_L)$. Furthermore, (IC_H) gives that $w_H^* - g(e_H^*, x_H) \geq w_L^* - g(e_L^o, x_H) > 0$, while (IC_L) gives that $w_H^* - g(e_H^*, x_L) \leq 0$. These two conditions together imply that $e_H^* \geq e_L^o > 0$. Because $\alpha_P = 1$ and the principal will be best off when the agent is of type x_L, her utility is $OWEU_P = \pi(e_L^o) - g(e_L^o, x_L)$.

Suppose instead that $\alpha_A \overline{p}_{H,A}(h) + (1 - \alpha_A)\underline{p}_{H,A}(h) = 1$. This can only hold if $\alpha_A = 1$ and $b = 1$. We then have that $\overline{p}_{H,P}(h) = 1$, which implies that the principal must be best off when the agent is of type x_H. Condition (B3) then implies that $e_H^* = e_H^o$, while (PC) implies that $w_H^* = g(e_H^o, x_H)$. Furthermore, (IC_H) gives that $0 \geq w_L^* - g(e_L^o, x_H)$, while (IC_L) gives that $w_L^* - g(e_L^*, x_L) \geq w_H^* - g(e_H^o, x_L)$. Together these imply that $e_L^* < e_H^o$. Because $\alpha_P = 1$ and the principal is best off when the agent is of type x_H, the principal's utility is $OWEU_P = \pi(e_H^o) - g(e_H^o, x_H)$.

Finally, if $\alpha_A \overline{p}_{H,A}(h) + (1 - \alpha_A)\underline{p}_{H,A}(h) \in (0, 1)$, we must have a corner contract. These are considered in Appendix B.

Case 4: $\lambda_\mathbf{H} > \mathbf{0}$ and $\lambda_\mathbf{L} > \mathbf{0}$. This case leads to a contradiction, by an argument similar to the one for Case 4 when $\alpha_P = 0$.

Corner contracts: $(\mathbf{w_H}, \mathbf{e_H}, \mathbf{w_L}, \mathbf{e_L})$ for which $\pi(\mathbf{e_H}) - \mathbf{w_H} = \pi(\mathbf{e_L}) - \mathbf{w_L}$. These are analyzed in Appendix B and give the principal the utility in (B19).

I now have to establish which type of contract is optimal. Suppose that $a > 0$. The situation when $a = 0$ is considered below. The contracts in Cases 1 and 2 both dominate the corner contracts. Case 3 gives the principal the same utility as the corner contracts. The same argument as was used in Appendix B for $\alpha_P \in (0,1)$ works to show that there exists $\hat{\alpha} \in (0,1)$ such that the optimal contract has distortion at the bottom for all $\alpha_A < \hat{\alpha}$, has distortion at the top for all $\alpha_A > \hat{\alpha}$, and has either distortion at the top or distortion at the bottom when $\alpha_A = \alpha_P$.

Suppose now that $a = 0$. If $\alpha_A = 0$, Case 1 is ruled out, and Case 2 dominates Case 3 and the corner contracts. The optimal contract therefore has distortion at the bottom.

If, on the other hand, $\alpha_A > 0$, there does not exist a solution to the principal's problem when $a = 0$ and $\alpha_P = 1$. To see this, suppose first that the principal offers the contract $(\tilde{w}_H, e_H^o, \tilde{w}_L, e_L^o)$, where $\tilde{w}_H$ and $\tilde{w}_L$ are such that (PC) and (IC_L) both hold with equality. With this contract, she pays the type x_L agent a lower wage and the type x_H agent a higher wage than with a "sell the firm to the agent" contract. Thus, the principal is best off when the agent is of type x_L, and her utility equals $K + \alpha_A b[\pi(e_L^o) - g(e_L^o, x_L) - \pi(e_H^o) + g(e_H^o, x_L)]$, where K is her utility with a "sell the firm to the agent" contract.

Suppose now that the principal instead offers the contract $(\widehat{w}_H, \widehat{e}_H, \widehat{w}_L, e_L^o)$, where $\widehat{w}_L \equiv \tilde{w}_L - \gamma$ with $\gamma > 0$, and that the contract satisfies both of (PC) and (IC_L) with equality. That is,

$$\alpha_A b(\widehat{w}_H - g(\widehat{e}_H, x_H)) + (1 - \alpha_A b)(\widehat{w}_L - g(e_L^o, x_L)) = 0 \tag{C6}$$

and

$$\widehat{w}_L - g(e_L^o, x_L) = \widehat{w}_H - g(\widehat{e}_H, x_L). \tag{C7}$$

This contract has distortion at the top. Solving (C6) and (C7) for $\widehat{w}_L$ gives that $\widehat{w}_L = g(e_L^o, x_L) + \alpha_A b(g(\widehat{e}_H, x_H) - g(\widehat{e}_H, x_L))$. With distortion at the top the principal is best off when the agent is of type x_L, hence, since $\alpha_P = 1$, her utility with this contract is

$$OWEU_P = \pi(e_L^o) - \widehat{w}_L = \pi(e_L^o) - g(e_L^o, x_L) - \alpha_A b\left(g(\widehat{e}_H, x_H) - g(\widehat{e}_H, x_L)\right).$$

This is strictly increasing in $\widehat{e}_H$ for all $\widehat{e}_H > 0$. The principal therefore want to distort effort for the high-efficiency type towards infinity. It follows that no solution exists to the principal's problem when $\alpha_P = 1$, $a = 0$, and $\alpha_A > 0$.