The Economics of Contingent Re-Auctions

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I. Online Appendix

Supplementary Material for the Proof of Proposition 1. We prove existence first for the ascending-bid auction. To improve readability, for the purposes of this proof we use the notation d instead of d^A for the delay threshold. In order to prove that a symmetric equilibrium exists, define

$$V_l^i(l_i; d) = E_{l_{-i},h_{-i}}\left[\max\{l_i - p_l(l_{-i}, r_1), 0\} \mid l_j \leq d\left(h_j\right) \text{ each } j \neq i\right]$$

$$V_h^i(h_i; d) = E_{l_{-i},h_{-i}}\left[\max\{h_i - p_h(h_{-i}, r_2), 0\} \mid l_j \leq d\left(h_j\right) \text{ each } j \neq i\right].$$

Consider the space Ξ of all continuous functions $d : \left[\underline{h}, \bar{h}\right] \rightarrow \left[r_1, \bar{l}\left(\bar{h}\right)\right]$. Let the distance between two functions d and d' in Ξ be defined as

$$\text{dist}\left(d, d'\right) = \sup_{h\in\left[\underline{h},\bar{h}\right]} \left|d(h) - d'(h)\right|. \tag{1}$$

This space is bounded below by the constant function at $d(h) = r_1$ and above by the constant function at $d(h) = \bar{l}\left(\bar{h}\right)$.

Remember that we assumed that the derivative $\frac{d\bar{l}(h)}{dh}$ is bounded above, hence there exists a real number K such that $\frac{d\bar{l}(h)}{dh} \leq K$. Define

$$\overline{K} = \max\left\{K, \frac{1}{\Pr\left[l_i \leq r_1\right]}\right\}$$

and consider the subset $\Psi \subset \Xi$ such that, for each $d \in \Psi$ we have

$$\left|h - h'\right| \leq \varepsilon \qquad \rightarrow \qquad \left|d(h) - d\left(h'\right)\right| \leq \overline{K}\varepsilon.$$

The set Ψ is non-empty, closed, bounded and convex. Furthermore, the subset Ψ of functions is equicontinuous since, for each $\varepsilon > 0$ we can find $\delta = \frac{\varepsilon}{\overline{K}}$ such that $\left|h - h'\right| < \delta$ implies $\left|d(h) - d\left(h'\right)\right| < \varepsilon$ for each $d \in \Psi$.

For each h let

$$\chi(h; d) = \sup\left\{l \leq \bar{l}(h) \middle| V_l^i(l; d) < V_h^i(h; d)\right\}.$$

Define a mapping $\xi : \Xi \to \Xi$ as follows. Given $d \in \Xi$ the function $\xi(d)$ is defined as

$$\xi(d)(h_i) = \max\{r_1, \chi(h_i; d)\}$$

for each $h_i \in [\underline{h}, \bar{h}]$. The mapping ξ is continuous in the metric defined by (1). If we can prove that ξ maps Ψ into itself, i.e. $\xi(\Psi) \subset \Psi$ then we can apply the Schauder Fixed-Point Theorem and conclude that a fixed point d^* exists.

To see that $\xi(d) \in \Psi$ whenever $d \in \Psi$, observe that we only need to consider points h such that $\xi(d)(h) = \chi(h; d)$. If $\chi(h; d) = \bar{l}(h)$ then we are done, since by assumption $\frac{d\bar{l}(h)}{dh} \leq K \leq \overline{K}$. Thus, consider a point h at which there is an open neighborhood such that $\chi(h; d) < \bar{l}(h)$. By construction, at all such points we must have

$$V_l^i(\chi(x; d); d) = V_h^i(x; d) \tag{2}$$

for each $x \in (h_i - \varepsilon, h_i + \varepsilon)$.

From this point on, in order to keep the notation as simple as possible, we will consider only the case of a single opponent and a differentiable function χ. The argument can be extended without major changes to the case of multiple opponents and non-differentiable functions.

Since (2) holds for each point in $(h_i - \varepsilon, h_i + \varepsilon)$, whenever χ is differentiable we have

$$\chi'(h_i; d) = \frac{\frac{dV_h^i(h_i;d)}{dh}}{\frac{dV_l^i(\chi(h_i;d);d)}{dl}}.$$

With a single opponent we have

$$\begin{aligned} V_l^i(l_i; d) &= \int_{\underline{h}}^{\bar{h}} \int_{\underline{l}}^{l_i} \max\left\{l_i - p_l\left(l_j, r_1\right), 0\right\} f\left(l_j, h_j \mid l_j < d\left(h_j\right)\right) dl_j dh_j \\ &= \int_{\underline{h}}^{\bar{h}} \int_{\underline{l}}^{l_i} \left(l_i - \max\left\{l_j, r_1\right\}\right) f\left(l_j, h_j \mid l_j < d\left(h_j\right)\right) dl_j dh_j, \end{aligned}$$

so that

$$\begin{aligned} \frac{dV_l^i(l_i; d)}{dl} &= \int_{\underline{h}}^{\bar{h}} \int_{\underline{l}}^{l_i} f\left(l_j, h_j \mid l_j < d\left(h_j\right)\right) dl_j dh_j \\ &= \frac{\int_{\underline{h}}^{\bar{h}} \int_{\underline{l}}^{\min\{l_i, d(h_j)\}} f\left(l_j, h_j\right) dl_j dh_j}{\Pr\left[l_j < d\left(h_j\right)\right]}. \end{aligned}$$

Since we are looking at the point $x = \chi(h_i; d)$ we have

$$\left.\frac{dV_l^i(l_i; d)}{dl}\right|_{l_i = \chi(h_i; d)} = \frac{\Pr\left[l_j \leq \min\left\{\chi(h_i; d), d\left(h_j\right)\right\}\right]}{\Pr\left[l_j < d\left(h_j\right)\right]} \geq \frac{\Pr\left[l_j \leq r_1\right]}{\Pr\left[l_j < d\left(h_j\right)\right]}$$

On the other hand,

$$V_h^i(h_i; d) = \int_{\underline{h}}^{h_i} \int_{\underline{l}}^{\bar{l}(\overline{h})} \max\{h_i - \max\{h_j, r_2\}\} f\left(l_j, h_j \mid l_j < d\left(h_j\right)\right) dl_j dh_j$$

so that

$$\frac{dV_h^i(h_i; d)}{dh_i} = \frac{\int_{\underline{h}}^{h_i} \int_{\underline{l}}^{d(h_j)} f\left(l_j, h_j\right) dl_j dh_j}{\Pr\left[l_j < d\left(h_j\right)\right]} \leq \frac{1}{\Pr\left[l_j < d\left(h_j\right)\right]}.$$

We conclude that the derivative of

$$\chi'(h_i; d) = \frac{\frac{dV_h^i(h_i;d)}{dh}}{\frac{dV_l^i(\chi(h_i;d);d)}{dl}} \leq \frac{\frac{1}{\Pr\left[l_j < d\left(h_j\right)\right]}}{\frac{\Pr\left[l_j \leq r_1\right]}{\Pr\left[l_j < d\left(h_j\right)\right]}} = \frac{1}{\Pr\left[l \leq r_1\right]} \leq \overline{K}.$$

This completes the proof.