Online Appendix to 'Personal Influence': Social Context and Political Competition

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This appendix contains the proofs of Proposition 1, Proposition 2, Proposition 3, and Proposition 4. Furthermore, we provide an additional result in which we show that the equilibria described in Proposition 3 are also equilibria of a more general model in which parties can target political advertising to different groups of voters. This result is contained in Proposition 5.

Proof Proposition 1.

First Step: Symmetric Pure Strategy Equilibria. We first consider pure-strategy equilibria. Let $s^* = (s_L^*, s_R^*)$ be part of a symmetric pure-strategy equilibrium. Then s_j^* prescribes to select candidate $t^* \in T$ with probability 1, and to advertise with intensity $x^*(t^*)$, for all $j \in \{L, R\}$. Note that in any pure-strategy equilibrium, it has to be that $x^*(t^*) = 0$. Hence, $U_L(s^*|t^*) = -(1-m)/2$. There are two possibilities, which we now analyze.

Consider the case of $t^* = m$, and let party L deviate by selecting $t_L = e$; in this case the best advertising strategy is $x_L(e) = 0$, and denoting this strategy by $\tilde{s}_L$, we have that $U_L(\tilde{s}_L, s_R^*|e) = -(2-3m)/4 > U_L(s^*|m)$, which contradicts our hypothesis that s^* is an equilibrium.

Consider now the case of $t^* = e$, and let party L deviate by selecting $t_L = m$ and $x_L(m)$; denote this strategy by $\tilde{s}_L$. Observe that such deviation is profitable only if $x_L(m) \neq 0$. Thus, assume that $x_L(m) > 0$. We now derive the optimal advertising level given s_R^*, which we denote by $x_L^*(m)$. To do this, we start by observing that:

$$\mu_L^*(\tilde{s}_L, s_R^*|m, e) = \frac{1}{2} + \frac{m}{4} - \frac{m}{4}(1 - x_L(m))^{k+1},$$

and it is readily seen that $\mu_L^*(\tilde{s}_L, s_R^*|m, e) \in (1/2, 1/2+m/4)$, for all $x_L(m) \in (0,1)$, which implies that $\pi_L(\tilde{s}_L, s_R^*|m, e) \in (0,1)$, for all $x_L(m) \in (0,1)$. Next, since $\pi_L(\tilde{s}_L, s_R^*|m, e) \in (0,1)$, it follows that party L's expected utility of playing $\tilde{s}_L$ against s_R^* is

$$U_L(\tilde{s}_L, s_R^*|m) = \left(\frac{5 - (1 - x_L(m))^{k+1}}{8}\right)\left(1 - \frac{3}{2}m\right) - (1-m) - \alpha x_L(m),$$

which is concave in $x_L(m)$. Hence, the optimal $x_L^*(m) \in (0,1)$ solves

$$(k+1)(1 - x_L^*(m))^k = \frac{16\alpha}{2-3m}. \tag{7}$$

Note that $x_L^*(m)$ is decreasing in α, and $x_L^*(m) \geq 0$ if and only if $\alpha \leq (2-3m)(k+1)/16$, and $x_L^*(m) = 1$ if and only if $\alpha = 0$. Thus, if $\alpha \geq (2-3m)(k+1)/16$, then $x_L^*(m) = 0$ and a pure-strategy equilibrium exists. If instead $\alpha < (2-3m)(k+1)/16$, party L will not deviate from s_L^* if and only if $U_L(s^*|e) \geq U_L(\tilde{s}_L, s_R^*|m)$, where abusing notation $\tilde{s}_L$ prescribes to advertise a moderate candidate with intensity $x_L^*(m)$. The latter inequality is satisfied if and only if

$$(8) \qquad (1 - x_L^*(m))^{k+1} \geq \frac{2 - 7m - 16\alpha x_L^*(m)}{2 - 3m}.$$

Defining $p = 1 - x$ and substituting (7) into (8), we have that $U_L(s^*|e) \geq U_L(\tilde{s}_L, s_R^*|m)$ if and only if

$$(9) \qquad \alpha \geq \alpha^*(k) \equiv \frac{(2 - 7m)(k + 1)}{16(k + 1 - k\underline{p})},$$

where $\underline{p}$ is the unique solution to

$$(k + 1)\underline{p}^k = \frac{16\alpha}{2 - 3m},$$

and $\alpha^*(k) < (2 - 3m)(k + 1)/16$ for all $p \in (0, 1)$. Putting together these observations it follows that a pure-strategy equilibrium exists if and only if $\alpha > \alpha^*(k)$, and in a pure-strategy equilibrium parties always select extreme candidates and they never advertise.
Second Step: Symmetric Mixed Strategy Equilibria. We now consider symmetric mixed-strategy equilibria. For convenience we use the notation $\sigma \equiv \sigma(e)$. Assume that a symmetric mixed-strategy equilibrium exists, and let $s_j^* = (\sigma^*, x^*(t))$, $j \in \{L, R\}$, denote the equilibrium strategy profile. Note that in equilibrium $x^*(e) = 0$. Consider a profile $s = (s_L, s_R)$ with $x_j(e) = 0$, $j \in \{L, R\}$; under this profile we have that:

$$U_L(s|e) = \sigma_R(1 - m)(\pi_L(s|e, e) - 1) + (1 - \sigma_R)\left(1 - \frac{3m}{2}\right)(\pi(s|e, m) - 1),$$

and

$$U_L(s|m) = \sigma_R\left(\pi_L(s|m, e)\left(1 - \frac{3m}{2}\right) - (1 - m)\right) + \\ + (1 - \sigma_R)\left(\pi_L(s|m, m)(1 - 2m) - \left(1 - \frac{3m}{2}\right)\right) - \alpha x_L(m).$$

Moreover,

$$\frac{\partial \pi_L(s|m,e)}{\partial x_L(m)} = \frac{\partial \pi_L(s|m,m)}{\partial x_L(m)} =$$
$$= \frac{1}{8}(k+1)(1-x_L(m))^k \rho_L(e|\emptyset, s, k),$$

and $U_L(s|m)$ is concave in $x_L(m)$. Hence, in a symmetric equilibrium, $\frac{\partial U_L(s|m)}{\partial x_L(m)}|_{s^*} = 0$ if and only if:

$$(k+1)(1-x^*(m))^k \rho(e|\emptyset, s^*, k) = \frac{16\alpha}{2-4m+\sigma m}, \tag{10}$$

where

$$\rho(e|\emptyset, s^*, k) = \frac{\sigma}{\sigma + (1-\sigma)(1-x^*(m))^{k+1}}.$$

Next, in a symmetric mixed-strategy political equilibrium each party is indifferent between selecting a moderate candidate and selecting an extremist candidate, i.e., $U_L(s^*|e) = U_L(s^*|m)$. Since in a symmetric equilibrium we have that

$$\pi(s^*|e,e) = \pi(s^*|m,m) = \frac{1}{2}$$
$$\pi(s^*|e,m) = \frac{1}{2} - \frac{1}{8}\rho(e|\emptyset, s^*, k)(1-(1-x^*(m))^{k+1}) = 1 - \pi(s^*|m,e),$$

it follows that $U_L(s^*|e) = U_L(s^*|m)$ if and only if:

$$\rho(e|\emptyset, s^*, k)(1-(1-x^*(m))^{k+1}) = \frac{4m + 16\alpha x^*(m)}{2-3m}. \tag{11}$$

Note that condition (11) is equivalent to condition (2) stated in Proposition 1.

We now show existence of a symmetric mixed-strategy equilibrium. We first investigate the equilibrium condition (10). Define

$$f(\sigma, p) = \frac{(k+1)p^k \sigma(2-4m+\sigma m)}{\sigma + (1-\sigma)p^{k+1}},$$

and note that the equilibrium condition (10) holds if and only if (σ, p) are such that $f(\sigma, p) = 16\alpha$. The following properties of $f(\cdot,\cdot)$ will prove useful for the proof.

Property 1: $f(0,p) = 0$ for all $p \in (0,1)$;

Property 2: $f(1,p) = (k+1)p^k(2-3m)$, which is increasing in p;

Property 3:

$$\frac{\partial f(\sigma,p)}{\partial \sigma} = \frac{(k+1)p^k}{(\sigma + (1-\sigma)p^{k+1})^2}\left(p^{k+1}(2-4m+m\sigma) + m\sigma(\sigma + (1-\sigma)p^{k+1})\right) > 0.$$

Properties 1, 2, and 3 imply that $\widetilde{\sigma}(p) : f(\tilde{\sigma}, p) = 16\alpha$ is a well defined function of p for all $p \in [\underline{p}, 1]$, where $\underline{p}$ solves $f(1, \underline{p}) = 16\alpha$, i.e.,

$$(k+1)\underline{p}^k = \frac{16\alpha}{2-3m}. \tag{12}$$

Note that $\underline{p} \in (0,1)$ if and only if $\alpha < (2-3m)(k+1)/16$.

We now study how $\widetilde{\sigma}(p)$ behaves in $p \in [\underline{p}, 1]$. The following properties of $\widetilde{\sigma}(\cdot)$ are useful:

Property 4: $\widetilde{\sigma}(\underline{p}) = 1$;

Property 5: $\widetilde{\sigma}(1) \in (0,1)$ solves

$$\widetilde{\sigma}(1)(2-4m+\widetilde{\sigma}(1)m) = \frac{16\alpha}{k+1};$$

Property 6: $\partial f(p,\sigma)/\partial p$ may change sign only once, and

$$\frac{\partial f(\sigma,p)}{\partial p}|_{\underline{p},\widetilde{\sigma}(\underline{p})} > 0.$$

Note that Property 6 follows from Property 4 and inspection of

$$\frac{\partial f(\sigma,p)}{\partial p} = \frac{(k+1)p^{k-1}\sigma(2-4m+\sigma m)(k\sigma - p^{k+1}(1-\sigma))}{[\sigma+(1-\sigma)p^{k+1}]^2}.$$

Using the implicit function theorem and invoking properties 1-6, we can summarize the results in the following claim:

Claim 1. For all $\alpha < (2-3m)(k+1)/16$ the following holds: $\tilde{\sigma}(\underline{p}) = 1$, $\tilde{\sigma}(p) \in (0,1)$ for all $p \in (\underline{p}, 1)$ and there exists a $v \in (\underline{p}, 1]$ such that $\frac{\partial \tilde{\sigma}(p)}{\partial p} < 0$ for all $p \in [\underline{p}, v]$, while $\frac{\partial \tilde{\sigma}(p)}{\partial p} > 0$ for all $p \in (v, 1]$, where $\underline{p}$ is the solution to (12). If $\alpha > (2-3m)(k+1)/16$ an equilibrium in mixed strategies does not exist.

We now consider the indifference equilibrium condition. Here, note that the equilibrium condition (11) can be rewritten as follows:

$$\bar{\sigma}(p) = \frac{p^{k+1}(4m+16\alpha(1-p))}{(1-p^{k+1})(2-7m-16\alpha(1-p))}. \tag{13}$$

Furthermore,

$$\frac{\partial \bar{\sigma}(p)}{\partial p} = \frac{p^k\left(\frac{(k+1)(4m+16\alpha(1-p))}{1-p^{k+1}} - \frac{16\alpha(2-3m)p}{2-7m-16\alpha(1-p)}\right)}{(1-p^{k+1})(2-7m-16\alpha(1-p))}. \tag{14}$$

Case I. Assume that $2-7m-16\alpha > 0$, which is equivalent to $\alpha < (2-7m)/16$.

Given this, we have that $\bar{\sigma}(0) = 0$, $\bar{\sigma}(1) = \infty$. Moreover, $\frac{\partial \bar{\sigma}(p)}{\partial p} > 0$ for all $p \in (0,1)$.

To see this note that $\frac{\partial \bar{\sigma}(p)}{\partial p} > 0$ if and only if

$$\frac{(k+1)(4m+16\alpha(1-p))}{1-p^{k+1}} - \frac{16\alpha(2-3m)p}{2-7m-16\alpha(1-p)} > 0.$$

Since the LHS is increasing in k, it is sufficient to show that:

$$\frac{(4m+16\alpha(1-p))}{1-p} - \frac{16\alpha(2-3m)p}{2-7m-16\alpha(1-p)} > 0,$$

that is

$$4m(2-7m-16\alpha(1-p)) + 16\alpha(1-p)(2-7m-16\alpha(1-p)-p(2-3m)) > 0,$$

which is equivalent to

$$B(p) = 4m(2-7m) + 16\alpha(1-p)((1-p)(2-3m-16\alpha)-8m) > 0.$$

Note that:

$$\frac{\partial B(p)}{\partial p} = -32\alpha((1-p)(2-3m-16\alpha)-4m),$$

and $\frac{\partial B(p)}{\partial p}|_{p=0} = -32\alpha(2-7m-16\alpha) < 0$ where the inequality follows because we are assuming that $2-7m-16\alpha > 0$, while $\frac{\partial B(p)}{\partial p}|_{p=1} > 0$. Given these two observations and the fact that $B(p)$ is convex in p, it follows that $B(p)$ is minimized whenever p is such that $(1-p)(2-3m-16\alpha) = 4m$. In this case $B(p)$ equals $4m(2-7m-16\alpha(1-p)) > 0$, where the inequality follows from $2-7m-16\alpha > 0$. This proves that $\frac{\partial \bar{\sigma}(p)}{\partial p} > 0$ for all $p \in (0,1)$.

Since $\bar{\sigma}(0) = 0$, $\bar{\sigma}(1) = \infty$ and $\bar{\sigma}(p)$ is increasing in p, there exists a unique $\bar{p} \in (0,1)$ such that $\bar{\sigma}(\bar{p}) = 1$, which is given by

$$\bar{p}^{k+1} = \frac{2-7m-16\alpha(1-\bar{p})}{2-3m}. \tag{15}$$

We summarize these observations in the following claim:

Claim 2. For all $\alpha < (2-7m)/16$ the following holds: $\bar{\sigma}(0) = 0$, $\bar{\sigma}(\bar{p}) = 1$, $\bar{\sigma}(p)$ is increasing for all $p \in [0,1]$, and $\bar{p}$ is the solution to (15).

Combining Claim 1 and Claim 2 it follows that for all $\alpha < (2-7m)/16$ a mixed-strategy equilibrium exists if and only if $\bar{p} > \underline{p}$, which holds if and only if $\alpha < \alpha^*(k)$, where $\alpha^*(k)$ is defined as in (9). Since $\alpha^*(k) > (2-7m)/16$ and by assumption $\alpha < (2-7m)/16$, we conclude that for all $\alpha < (2-7m)/16$ a symmetric mixed-strategy equilibrium exists.

Case II. Assume that $2-7m-16\alpha < 0$, which is equivalent to $\alpha \geq (2-7m)/16$.

Define $\hat{p}$ such that $2-7m-16\alpha(1-\hat{p}) = 0$, that is $\hat{p} = 1-(2-7m)/(16\alpha) \in (0,1)$.

Note that for all $p \in (0, \hat{p})$, (13) implies that $\bar{\sigma}(p) < 0$. Hence, a necessary condition for an equilibrium is that $p \in [\hat{p}, 1)$.

Let $p \in [\hat{p}, 1)$; note that for all $p \in [\hat{p}, 1)$, the following holds $\bar{\sigma}(p) > 0$, $\bar{\sigma}(\hat{p}) = \infty$ and $\bar{\sigma}(1) = \infty$. Furthermore, there exists a $p = z$ such that $\bar{\sigma}(z) = 1$ if and only if

$$\psi(z, \alpha) = z^{k+1} - \frac{2 - 7m - 16\alpha(1 - z)}{2 - 3m} = 0.$$

Moreover, $\psi(\hat{p}, \alpha) = \hat{p}^{k+1} > 0$, $\psi(1, \alpha) > 0$, and

$$\frac{\partial \psi(z, \alpha)}{\partial z} = (k + 1)z^k - \frac{16\alpha}{2 - 3m},$$

where $\psi(z, \alpha)$ is convex in z. Recall from equation (12) that $\underline{p}$ is such that $\underline{p}^k = 16\alpha/((2 - 3m)(k + 1))$. Hence, if $\hat{p} > \underline{p}$, then $\frac{\partial \psi(z,\alpha)}{\partial z} > 0$ for all $z \in (\hat{p}, 1)$, which implies that $\bar{\sigma}(p) > 1$ for all $p \in (\hat{p}, 1)$. Hence, for existence of an equilibrium in mixed strategies it must be the case that $\hat{p} < \underline{p}$.

Suppose that $\hat{p} < \underline{p}$. Then, $\psi(z, \alpha)$ is minimized when $z = \underline{p}$, and a necessary condition for existence of a mixed-strategy equilibrium is that $\bar{\sigma}(p) < 1$ for some p, which requires that

$$\psi(\underline{p}, \alpha) = \frac{16\alpha\underline{p}}{(2 - 3m)(k + 1)} - \frac{2 - 7m - 16\alpha(1 - \underline{p})}{2 - 3m} < 0,$$

which is true if and only if

$$\alpha < \frac{(2 - 7m)(k + 1)}{16(k + 1 - k\underline{p})} = \alpha^*(k).$$

Moreover, $\hat{p} < \underline{p}$ if and only if $1 - \frac{2-7m}{16\alpha} < \underline{p}$, which is always true when $\alpha \leq \frac{(2-7m)(k+1)}{16(k+1-k\underline{p})} = \alpha^*(k)$.

We summarize these observations in the following claim:

Claim 3. Suppose that $\alpha \in \left((2 - 7m)/16, \frac{(2-7m)(k+1)}{16(k+1-k\underline{p})}\right)$ then the following holds: there exists $(p_1, \bar{p})$ such that (i) $\hat{p} < p_1 < \underline{p} < \bar{p} < 1$ and (ii) $\bar{\sigma}(p_1) = \bar{\sigma}(\bar{p}) = 1$ and $\bar{\sigma}(p) \in (0, 1)$ for all $p \in [p_1, \bar{p}]$.

Combining Claim 1 and Claim 3 it follows that if $\alpha \in ((2 - 7m)/16, \alpha^*(k))$ a mixed-strategy equilibrium exists. Finally, combining Case 1 together with Case 2 it follows that a mixed-strategy equilibrium exists if and only if $\alpha < \alpha^*(k)$.

Final Step: Uniqueness. If $\alpha > \alpha^*(k)$ we know that the equilibrium is in pure strategies and it is unique. So, let $\alpha < \alpha^*(k)$. In this case the equilibrium is in mixed strategies and recall that at the equilibrium $s^* = (\sigma^*, p^*)$, conditions (10) and (11) are mutually satisfied. To prove that there exists a unique equilibrium it is sufficient to prove that at p^* the following is true: (i) $\frac{\partial \bar{\sigma}(p)}{\partial p}|_{p^*} > 0$ and (ii) $\frac{\partial \bar{\sigma}(p)}{\partial p}|_{p^*} < 0$. We now show that these two conditions hold.

We first show that $\frac{\partial \bar{\sigma}(p)}{\partial p}|_{p^*} > 0$. Note that, using expression (14), it follows that

$\frac{\partial \bar{\sigma}(p)}{\partial p}|_{p^*} > 0$ if and only if

$$\frac{(k+1)(4m+16\alpha(1-p^*))}{1-p^{*k+1}} - \frac{16\alpha(2-3m)p^*}{2-7m-16\alpha(1-p^*)} > 0. \tag{16}$$

At the equilibrium s^* condition (11) holds, which implies that

$$\rho(e|\emptyset, s^*, k) = \frac{4m+16\alpha(1-p^*)}{(1-p^{*k+1})(2-3m)}, \tag{17}$$

and substituting this expression in the equilibrium condition (10), we obtain that in equilibrium

$$\frac{(k+1)(4m+16\alpha(1-p^*))}{1-p^{*k+1}} = \frac{16\alpha(2-3m)}{(2-4m+\sigma^* m)p^{*k}}.$$

Thus, we can rewrite inequality (16) as follows:

$$16\alpha(2-3m)\left(\frac{1}{p^{*k}(2-4m+\sigma^* m)} - \frac{p^*}{2-7m-16\alpha(1-p^*)}\right) > 0,$$

which is satisfied if and only if

$$C(p^*) = 2-7m-16\alpha(1-p^*) - p^{*k+1}(2-4m+\sigma^* m) > 0.$$

To show that the last inequality holds, we observe that

$$\begin{aligned} \frac{\partial C(p^*)}{\partial p^*} &= 16\alpha - (k+1)p^{*k}(2-4m+\sigma^* m) \\ &> 16\alpha - (k+1)\underline{p}(2-4m+\sigma^* m) \\ &> 16\alpha - (k+1)\underline{p}(2-3m) = 0, \end{aligned}$$

where the first inequality follows from $p^* > \underline{p}$, the second inequality follows from $\sigma^* \in (0,1)$, and the last equality follows from the definition of $\underline{p}$. Hence, it is sufficient to show that $C(\underline{p}) > 0$. Using the expression for $\underline{p}$ we obtain

$$C(\underline{p}) = 2-7m-16\alpha(1-\underline{p}) - \frac{16\alpha\underline{p}(2-4m+\sigma^* m)}{(k+1)(2-3m)} > 0$$

whenever

$$(k+1)(2-3m)(2-7m-16\alpha(1-\underline{p})) - 16\alpha\underline{p}(2-4m+\sigma^* m) > 0,$$

which holds because

$$\begin{aligned} & (k+1)(2-3m)(2-7m-16\alpha(1-\underline{p}))-16\alpha\underline{p}(2-4m+\sigma^*m) \\ > & (k+1)(2-3m)(2-7m-16\alpha(1-\underline{p}))-16\alpha\underline{p}(2-3m) \\ = & (2-3m)(k+1)(2-7m)-16\alpha(2-3m)((k+1-k\underline{p}))>0, \end{aligned}$$

where the first inequality follows from $\sigma^* < 1$, and the last inequality follows from $\alpha < (k+1)(2-7m)/(16(k+1-k\underline{p}))$. Hence, $\frac{\partial\tilde{\sigma}(p)}{\partial p}|_{p^*} > 0$.

We now show that $\frac{\partial\tilde{\sigma}(p)}{\partial p}|_{p^*} < 0$. To see this note that, using expression (17), the equilibrium condition (10) can be rewritten as follows:

$$(k+1)p^{*k}\frac{4m+16\alpha(1-p^*)}{(2-3m)(1-p^{*k+1})}=\frac{16\alpha}{2-4m+\sigma^*m}, \tag{18}$$

and to establish that $\frac{\partial\tilde{\sigma}(p)}{\partial p}|_{p^*} < 0$ it is sufficient to show that the LHS is increasing in p^*. Since $(k+1)p^{*k}$ is obviously increasing in p^*, we need to show that $\frac{4m+16\alpha(1-p^*)}{(2-3m)(1-p^{*k+1})}$ is increasing in p^*, which, taking the derivative with respect to p, holds if and only if

$$\frac{(k+1)p^{*k}(4m+16\alpha(1-p^*))}{1-p^{*k+1}}-16\alpha>0.$$

Using (18) we have that

$$\begin{aligned} & \frac{(k+1)p^{*k}(4m+16\alpha(1-p^*))}{1-p^{*k+1}}-16\alpha \\ = & 16\alpha\left(\frac{2-3m}{2-4m+\sigma^*m}-1\right) \\ = & \frac{16\alpha m(1-\sigma^*)}{2-4m+\sigma^*m}>0. \end{aligned}$$

Combining these observations, it follows that there exists a unique equilibrium.

To complete the proof of Proposition 1 we show that $\alpha^*(k)$ is increasing in k. Note that

$$\frac{d\alpha^*(k)}{dk}=\frac{\partial\alpha^*(k)}{\partial k}+\frac{\partial\alpha^*(k)}{\partial\underline{p}}\frac{d\underline{p}}{dk},$$

where

$$\frac{d\underline{p}}{dk}=-\frac{1}{k\underline{p}^{k-1}}\left[p^k\ln(p)+\frac{16\alpha}{(2-3m)(k+1)^2}\right]$$

Using this last expression, and developing $\frac{\partial\alpha^*(k)}{\partial k}$ and $\frac{\partial\alpha^*(k)}{\partial\underline{p}}$, we have that

$$\frac{d\alpha^*(k)}{dk}=\frac{(2-7m)\underline{p}}{16(k+1-k\underline{p})^2}\left[1-\underline{p}^{k-1}-(k+1)\ln(p)\right]>0.$$

This completes the proof of Proposition 1.

Proof of Proposition 2. Let $A(p,\alpha) = 4m+16\alpha(1-p)$, $B(p,\alpha) = 2-7m-16\alpha(1-p)$ and $C(\sigma(p)) = 2-4m+\sigma(p)m$. With some abuse of notation we shall write A instead of $A(p,\alpha)$, and similarly for B and C. Recall that in equilibrium:

$$\begin{aligned} f(\sigma(p),p) - 16\alpha &= \frac{(k+1)p^k AC}{(2-3m)(1-p^{k+1})} = 0 \\ \sigma(p) &= \frac{Ap^{k+1}}{B(1-p^{k+1})}. \end{aligned}$$

We first prove the first part of the proposition, i.e., $x^*_{k+1}(m) < x^*_k(m)$, $\Psi(s^*_{k+1}, k+1) > \Psi(s^*_k, k)$ and $\pi(s^*_{k+1}|e,m) > \pi(s^*_k|e,m)$. To see this note that:

$$\begin{aligned} \frac{\partial f(\sigma(p),p)}{\partial k} &= \frac{Ap^k\left(C(1-p^{k+1}+(k+1)\ln(p)) + (k+1)m(1-p^{k+1})\frac{\partial\sigma(p)}{\partial k}\right)}{(2-3m)(1-p^{k+1})^2} \\ \frac{\partial\sigma(p)}{\partial k} &= \frac{Ap^{k+1}}{B(1-p^{k+1})^2}\ln(p). \end{aligned}$$

Clearly, $\frac{\partial\sigma(p)}{\partial k} < 0$, and therefore to show that $\frac{\partial f(\sigma(p),p)}{\partial k} < 0$ it is sufficient to note that $(1-p^{k+1}+(k+1)\ln(p)) < 0$ for all $p \in (0,1)$. Further, since the LHS of equation (18) is increasing in p^* and the RHS is decreasing in p^* since $\frac{\partial\bar{\sigma}(p)}{\partial p}|_{p^*} > 0$, it follows that in equilibrium $\frac{\partial f(\sigma(p),p)}{\partial p} > 0$ (recall that equation (18) is equivalent to equation (10)). Combining these two observations it follows that for all $\alpha < \alpha^*(k)$, an increase in k increases p, i.e., $x^*_{k+1} < x^*_k$. By investigation of equilibrium condition (2) in the paper, it is immediate to see that if $x^*_{k+1} < x^*_k$, then $1-\Psi(s^*_{k+1}, k+1) < 1-\Psi(s^*_k, k)$, which implies that $\Psi(s^*_{k+1}, k+1) > \Psi(s^*_k, k)$. Moreover, since $\pi(s^*_k|e,m) = 1/2-(1-\Psi(s^*_k,k))/8$, it follows that $\pi(s^*_{k+1}|e,m) > \pi(s^*_k|e,m)$.

We now prove the second part of the proposition, i.e., for sufficiently low α, if k increases then σ increases and polarization increases. First, note that:

$$\frac{d\sigma(p)}{dk} = \frac{\partial f(\sigma(p),p)}{\partial p}\frac{\partial\sigma(p)}{\partial k} - \frac{\partial f(\sigma(p),p)}{\partial k}\frac{\partial\sigma(p)}{\partial p},$$

where

$$\begin{aligned} \frac{\partial f(\sigma(p),p)}{\partial p} &= \frac{(k+1)p^{k-1}\left(AC(k+p^{k+1}) + p(1-p^{k+1})Am\frac{\partial\sigma(p)}{\partial p} - 16C\alpha p(1-p^{k+1})\right)}{(2-3m)(1-p^{k+1})^2} \\ \frac{\partial\sigma(p)}{\partial p} &= \frac{p^k}{B(1-p^{k+1})}\left(\frac{A(k+1)}{1-p^{k+1}} - \frac{16\alpha p(2-3m)}{B}\right). \end{aligned}$$

Using these expressions and noting that as $\alpha \to 0$ then $p \to 0$, and A, B, and C are bounded, it follows that:

$$\lim_{\alpha\to 0}\frac{d\sigma(p)}{dk}=\lim_{\alpha\to 0}\left(\frac{A^2C(k+1)}{B(2-3m)}\right)\lim_{\alpha\to 0}\left(-p^{2k}\ln(p)\right)=0^+.$$

Hence, there exists a $\hat{\alpha}(k)\in(0,\alpha^*(k))$ such that for all $\alpha\in(0,\hat{\alpha}(k))$ an increase in k leads to an increase in σ.

We finally show that for sufficiently small α,

$$\Pi(s^*,k)=\sigma^{*2}+2\sigma^*(1-\sigma^*)\pi(s^*|e,m),$$

is increasing in k. Note that for small α, $\sigma^*<1/2$ and therefore $\Pi(s^*,k)$ is increasing in σ, keeping constant $\pi(s^*|e,m)$. Since an increase in k leads to an increase in $\pi(s^*|e,m)$, the result follows. This concludes the proof of the proposition.

In what follows, with some abuse of notation, we shall denote by $i_{j,in}$ the indifferent group-j voter that has sampled only voters belonging to group j, $j=l,r$; analogously, $i_{j,out}$ is the indifferent group-j voter that has sampled at least one voter in group j', where $j,j'=l,r$ and $j\neq j'$.

Proof of Proposition 3. We start by considering symmetric pure strategy equilibria. Note that in equilibrium a party never advertises an extremist candidate. Also note that a (pure) strategy profile in which each party selects a moderate and does not advertise cannot be part of equilibrium; for otherwise, a party, by switching to an extremist candidate, would win the election with the same probability and would obtain a higher expected payoff, which contradicts optimality. We are then left with two possible candidates: (1.) parties select extremist candidates and do not advertise and (2.) parties select moderates and do advertise. We now analyze these two possibilities.
(1.) Consider a strategy profile s^* such that $\sigma=1$ and $x=0$. In this case the utility of party L is $U_L^*(s^*)=-(1-m)/2$. It is easy to check that the best deviation of party L is to select a moderate and to advertise. Let s_L be such strategy profile. The utility of party L is then

$$U_L(s_L,s_R^*)=\pi_L(s_L,s_R^*|m,e)\left(1-\frac{3m}{2}\right)-(1-m)-\alpha.$$

We now derive $\pi_L(s_L,s_R^*|m,e)$. First, all group-l voters observe that $t_L=m$, and, regardless of the realization of the sampling, they believe that $t_R=e$. Therefore, $i_{l,in}=i_{l,out}=1/2+m/4$. Second, all group-$r$ voters believe that $t_R=e$, a fraction β always samples group-r voters so that they believe that $t_L=e$, while the remaining group-r voters have sampled at least one voter in group l and therefore they know that $t_L=m$. Thus, $i_{r,in}=1/2$, while $i_{r,out}=1/2+m/4=i_{l,in}$. Putting together these facts it follows that for all $\mu>i_{l,in}=i_{r,out}$ party L never wins, and that for all $\mu\leq i_{l,in}$ party L wins with probability 1. Hence, $\pi_L(s_L,s_R^*|m,e)=\Pr[\mu\leq i_{l,in}]=5/8$. Therefore, s^* is equilibrium if and only if

$$U_L(s_L, s_R^*) = \frac{5}{8}\left(1 - \frac{3m}{2}\right) - (1-m) - \alpha \leq U_L^*(s^*) = -(1-m)/2,$$

which is satisfied if and only if

$$\alpha \geq \frac{2-7m}{16} = \bar{\alpha}.$$

(2.) Next, consider a strategy profile s^* such that $\sigma = 0$ and $x = 1$. In this case $U_L^*(s^*) = -(1-m)/2 - \alpha$. Suppose that if a group-l voter does not observe the ads from party L he believes that $t_L = e$; analogously for group-r voters. Clearly, the best deviation of party L is to select an extremist candidate and do not advertise. Let s_L be such strategy. The utility of party L is

$$U_L(s_L, s_R^*) = \left(1 - \frac{3m}{2}\right)[\pi_L(s_L, s_R^*|e, m) - 1].$$

We now derive $\pi_L(s_L, s_R^*|e, m)$. First, all group-l voters observe that party L does not advertise and therefore they believe that $t_L = e$; also, regardless of the sampling, all group-l voters believe that $t_R = m$. Hence, $i_{l,in} = i_{l,out} = 1/2 - m/4$. Second, all group-$r$ voters believe that $t_R = m$, a fraction β of group-r voters believe that $t_L = m$, while the remaining voters believe that $t_L = e$. Hence, $i_{r,in} = 1/2$, while $i_{r,out} = 1/2 - m/4 = i_{l,in}$. Using these considerations, three facts follow. One, for all $\mu \geq i_{r,in}$ party L never wins; two, for all $\mu \leq i_{r,out}$ party L wins with probability 1. Three, for all $\mu \in [i_{r,out}, i_{r,in}]$, total votes for party L are

$$TV_L = \frac{1}{2} + \frac{1+\beta}{2\tau}\left(\frac{1}{2} - \mu\right) - \frac{1}{\tau}\frac{m}{8},$$

and $TV_L > 1/2$ if and only if

$$\mu \leq \frac{1}{2} - \frac{m}{4}\frac{1}{1+\beta} = \mu^*,$$

where it is easy to check that $\mu^* \in [i_{r,out}, i_{r,in}]$. Combining these three facts we have that

$$\pi_L(s_L, s_R^*|e, m) = \Pr\left[\mu \leq \frac{1}{2} - \frac{m}{4}\frac{1}{1+\beta}\right] = \frac{3+4\beta}{8(1+\beta)}.$$

Hence, s^* is an equilibrium if and only if

$$U_L(s_L, s_R^*) = \left(1 - \frac{3m}{2}\right)\left(\frac{3+4\beta}{8(1+\beta)} - 1\right) \leq -\frac{1-m}{2} - \alpha,$$

which is satisfied if and only if

$$\alpha \leq \frac{2-7m-4m\beta}{16(1+\beta)} = \underline{\alpha}(\beta).$$

It is easy to verify that $\underline{\alpha}(\beta) < \bar{\alpha}$ and that $\underline{\alpha}(\beta) \geq 0$ if and only if $\beta \leq (2-7m)/(4m)$.

We now turn to symmetric mixed-strategy equilibria. Clearly, the only candidate is a strategy profile s^* in which each party selects an extremist candidate with probability $\sigma \in (0,1)$ and advertises only moderates. Randomization implies that a party is indifferent between selecting an extremist and a moderate, which holds if and only if:

$$\pi_L(s^*|m,e) = \frac{1-m+2\alpha}{2-3m}. \tag{19}$$

We now derive the expression for $\pi_L(s^*|m,e)$. Suppose that $t_L = m$ and $t_R = e$. First, a fraction β of group-l voters believe that $t_L = m$ and, with probability σ, that $t_R = e$. The remaining fraction $1-\beta$ believe that $t_L = m$ and that $t_R = e$. Hence, $i_{l,in} = 1/2 + m\sigma/4$, while $i_{l,out} = 1/2 + m/4$. Second, all group-r voters believe that $t_R = e$, a fraction β believe that $t_L = e$ with probability σ, and a fraction $1-\beta$ believe that $t_L = m$. Hence, $i_{r,in} = 1/2 + m(1-\sigma)/4$, while $i_{r,out} = 1/2 + m/4 = i_{l,out}$.

Given these observations there are two relevant cases to be considered: (1.) $\sigma \leq \frac{1}{2}$ and (2.) $\sigma \geq \frac{1}{2}$.

Case 1. Suppose $\sigma \leq 1/2$; then $i_{l,in} \leq i_{r,in} < i_{l,out} = i_{r,out}$. It follows that for all $\mu \leq i_{l,in}$ party L wins with probability 1. For all $\mu \in [i_{l,in}, i_{r,in}]$, total votes of party L are

$$TV_L = \frac{1}{2} + \frac{1+\beta}{2\tau}\left(\frac{1}{2} - \mu\right) + \frac{1}{2\tau}\frac{m}{4},$$

and $TV_L \geq 1/2$ if and only if

$$\mu \leq \frac{1}{2} + \frac{m}{4}\frac{1}{1+\beta} = \mu_1^*,$$

where $\mu_1^* \geq i_{l,in}$, and $\mu_1^* \leq i_{r,in}$ if and only if $\sigma \leq \beta/(1+\beta)$. Therefore, for all $\mu \in [i_{l,in}, i_{r,in}]$, if $\sigma \leq \beta/(1+\beta)$ party L wins with probability $\Pr(\mu \leq \mu_1^*) = (5+4\beta)\,/\,(8\,(1+\beta))$, while if $\sigma \geq \beta/(1+\beta)$ then party L wins with probability 1. For all $\mu \in [i_{r,in}, i_{l,out}]$, total votes of party L are

$$TV_L = \frac{1}{2} + \frac{1}{2\tau}\left(\frac{1}{2} - \mu\right) + \frac{1}{2\tau}\frac{m}{4}\left(1-\beta(1-\sigma)\right),$$

and $TV_L \geq 1/2$ if and only if

$$\mu \leq \frac{1}{2} + \frac{m}{4}\left(1-\beta(1-\sigma)\right) = \mu_2^*.$$

Clearly, $\mu_2^* \leq i_{l,out}$, and $\mu_2^* \geq i_{r,in}$ if and only if $\sigma \geq \beta/(1+\beta)$. Hence, for all $\mu \in [i_{r,in}, i_{l,out}]$, if $\sigma \leq \beta/(1+\beta)$, party L never wins, while if $\sigma \geq \beta/(1+\beta)$, party L wins with probability $\Pr(\mu \leq \mu_2^*) = (5 - \beta(1-\sigma))/8$. Finally, for all $\mu \geq i_{l,out}$ party L never wins. Summarizing, we have that: if $\sigma \leq \beta/(1+\beta)$ then $\pi_L(m, e|s^*) = (5+4\beta)/(8(1+\beta))$, while if $\sigma \in [\beta/(1+\beta), 1/2]$ then $\pi_L(m, e|s^*) = (5 - \beta(1-\sigma))/8$.

Case 2. Suppose $\sigma \geq 1/2$; then $i_{r,in} \leq i_{l,in} < i_{l,out} = i_{r,out}$. Here note that for all $\mu \leq i_{l,in}$ party L wins with probability 1. In contrast, for all $\mu \geq i_{l,out}$ party L never wins. Finally, for all $\mu \in [i_{l,in}, i_{l,out}]$, total votes of party L are

$$TV_L = \frac{1}{2} + \frac{1}{2\tau}\left(\frac{1}{2} - \mu\right) + \frac{1}{2\tau}\frac{m}{4}\left(1 - \beta(1-\sigma)\right),$$

and $TV_L \geq 1/2$ if and only if

$$\mu \leq \mu_2^*,$$

where it is easy to check that $\mu_2^* \in [i_{l,in}, i_{l,out}]$. Hence, for all $\mu \in [i_{l,in}, i_{l,out}]$, party L wins with probability $\Pr(\mu \leq \mu_2^*) = (5 - \beta(1-\sigma))/8$. Combining these observations it follows that if $\sigma \in [1/2, 1)$ then $\pi_L(s^*|m, e) = (5 - \beta(1-\sigma))/8$.

By combining case 1 and case 2, it follows that if $\sigma \in (0, \beta/(1+\beta)]$ then $\pi_L(s^*|m, e) = (5+4\beta)/(8(1+\beta))$, and if $\sigma \in [\beta/(1+\beta), 1)$ then $\pi_L(s^*|m, e) = (5 - \beta(1-\sigma))/8$.

Next, note that a mixed-strategy equilibrium exists only if $\sigma \in [\beta/(1+\beta), 1)$. Indeed, if $\sigma < \beta/(1+\beta)$, then $\pi_L(s^*|m, e)$ does not depend on σ. Therefore, the equilibrium condition (19) cannot be satisfied generically. Therefore, it must be that $\sigma \in [\beta/(1+\beta), 1)$, and the equilibrium condition (19) holds if and only if

$$\sigma^* = 1 - \frac{2 - 7m - 16\alpha}{\beta(2 - 3m)}.$$

Note that σ^* is increasing in α and, when $\alpha = \bar{\alpha}$ then $\sigma^* = 1$. Also if $\underline{\alpha}(\beta) \geq 0$, we have that at $\alpha = \underline{\alpha}(\beta)$, $\sigma^* = \beta/(1+\beta)$. If instead $\underline{\alpha}(\beta) \leq 0$ then as α approaches 0, σ^* converges to $1 - (2-7m)/(\beta(2-3m)) \geq \beta/(1+\beta)$, where the last inequality follows from the fact that $\underline{\alpha}(\beta) \leq 0$ (i.e., $\beta < (2-7m)/(4m)$). This concludes the proof of Proposition 3.

Proof of Proposition 4. It is immediate to see that an increase in β, increases the probability that a party selects an extremist candidate. We now show that an increase in β it increases the ex-ante expected probability that an extremist candidate wins the election. To see this note that:

$$\Pi(s^*) = [\sigma^*]^2 + 2\sigma^*(1-\sigma^*)\pi(s^*|e, m),$$

and therefore

$$\frac{d\Pi(s^*)}{d\beta} = 2\left((\sigma^* + \pi(s^*|e, m)(1 - 2\sigma^*))\frac{d\sigma^*}{d\beta} + \sigma^*(1-\sigma^*)\frac{d\pi(s^*|e, m)}{d\beta}\right).$$

Since in equilibrium $\pi_L(s^*|m,e) = (1-m+2\alpha)/(2-3m)$, and $\pi(s^*|e,m) = 1-\pi(s^*|m,e)$, it must be that $d\pi(s^*|e,m)/d\beta = 0$. Hence, it follows that:

$$\frac{d\Pi(s^*)}{d\beta} = 2\frac{d\sigma^*}{d\beta}\left(\sigma^* + \pi(s^*|e,m)(1-2\sigma^*)\right)$$
$$= 2\frac{d\sigma^*}{d\beta}\left(\sigma^*(1-\pi(s^*|e,m)) + \pi(s^*|e,m)(1-\sigma^*)\right) > 0.$$

This concludes the proof of Proposition 4.

We now consider a model where parties choose whether to advertise to group l, to group r, to both groups, or not to advertise. Abusing notation, assume that advertising to one group costs α, while advertising to both groups costs 2α. The following proposition shows that the equilibria described in Proposition 3 are also equilibria of this model.

PROPOSITION 5: *Suppose parties can target advertisement to group l, or group r, or to both groups. The following holds: (I.) If $\alpha > \bar{\alpha}$ there exists an equilibrium where parties select extremist candidates with probability one and they never advertise. (II.) If $\alpha \in (0, \max[0, \underline{\alpha}(\beta)])$ there exists an equilibrium in which parties select moderates and they only advertise to their closer group of independents. (III.) For every β, there exists $\alpha^* \in (\max[0, \underline{\alpha}(\beta)], \bar{\alpha})$, such that if $\alpha \in (\alpha^*, \bar{\alpha})$, in equilibrium parties select extremist candidates with probability σ^* and they advertise a moderate only to their closer group of independents, where σ^* is given by equation (6) in the paper.*

Proof of Proposition 5. Consider the strategy s^* in which parties select extremist candidates with probability one and they never advertise. From Proposition 3 we know that for all $\alpha > \bar{\alpha}$ the following deviation is not profitable: a party selects a moderate and advertises only to its own group. It is then sufficient to show that this is indeed the best deviation for a party. To see this, first note that if party L selects a moderate and advertises only to its own group, the probability of winning is $\Pr(\mu \leq 1/2 + m/4)$. Second, suppose that party L deviates by selecting a moderate and by advertising to group r only. It is immediate to check that in this case the probability of winning of party L will be lower than $\Pr(\mu \leq 1/2 + m/4)$. Therefore, the latter deviation is at least as good as the deviation considered above. The other possible deviation is one in which party L selects a moderate and advertises to both groups. Again, it is a matter of simple algebra to check that the probability of winning under this deviation cannot be higher than $\Pr(\mu \leq 1/2 + m/4)$. However, this deviation involves higher costs of advertising.

Consider now the strategy s^* in which parties select moderates and advertise to their own group. If $\underline{\alpha}(\beta) \leq 0$ then Proposition 3 implies that this is not an equilibrium. If instead $\underline{\alpha}(\beta) > 0$, Proposition 3 implies that for all $\alpha \leq \underline{\alpha}(\beta)$ the following deviation is not profitable: a party selects an extremist candidate and does not advertise. Moreover, if a party deviates from s^* by advertising to both groups, then simple algebra delivers that the party will face the same probability of winning as in s^*, and it will face an higher cost. Hence, we only need to check the following deviation s^d: party L selects a moderate and advertises only to group-r voters. However, note that s^* and s^d are cost equivalent, but the probability of winning of party L under s^d is lower than under s^*.

We now consider the mixed-strategy equilibrium defined in Proposition 3. We already know that for all $\alpha \in (\max[0, \underline{\alpha}(\beta)], \bar{\alpha})$ there exists a σ, defined by equation (6) in the paper, such that parties are indifferent between selecting an extremist and selecting a moderate which they advertise only to their closer group. Suppose party R follows this strategy s^*_R. We consider the possible deviations of party L.

Deviation 1: consider the following strategy s_L: party L selects a moderate and advertises to group r only. In order to derive $\pi_L(s_L, s^*_R|m, e)$, note that a fraction β of group-l voters believe that $t_L = e$ and, with probability σ, that $t_R = e$, while the remaining fraction of group-l voters believe that $t_L = m$ and $t_R = e$. Hence, $i_{l,in} = 1/2 - m(1-\sigma)/4$ and $i_{l,out} = 1/2 + m/4$. All group-r voters believe that $t_L = m$ and $t_R = e$, so that $i_{r,in} = i_{r,out} = 1/2 + m/4$. It follows that $\pi_L(s_L, s^*_R|m, e) = \Pr(\mu \leq 1/2 + (m/4)\,(1-\beta(1-\sigma))\,/\,(1+\beta)) < \pi_L(s^*|m, e) = \Pr(\mu \leq 1/2 + (m/4)\,(1-\beta(1-\sigma)))$. Next, we derive $\pi_L(s_L, s^*_R|m, m)$. In this case, a fraction β of group-l voters believe that $t_L = e$ and, with probability σ, that $t_R = e$. The remaining fraction of group-l voters believe that $t_L = t_R = m$. Hence $i_{l,in} = 1/2 - m(1-\sigma)/4$ and $i_{l,out} = 1/2$. All group-r voters believe that $t_L = t_R = m$ and therefore $i_{r,in} = i_{r,out} = 1/2$. It is now easy to check that $\pi_L(s_L, s^*_R|m, m) \leq 1/2 = \pi_L(s^*|m, m)$. Putting together these two facts, it follows that deviation 1 is not profitable.

Deviation 2: consider the following strategy s_L: party L selects a moderate and advertises to both groups. We first derive $\pi_L(s_L, s^*_R|m, e)$. A fraction β of group-l voters believe that $t_L = m$ and, with probability σ, that $t_R = e$. The remaining fraction believe that $t_L = m$ and $t_R = e$. Hence, $i_{l,in} = 1/2 + m\sigma/4$ and $i_{l,out} = 1/2 + m/4$. All group-r voters believe that $t_L = m$ and $t_R = e$; therefore $i_{r,in} = i_{r,out} = 1/2 + m/4$. It is now easy to check that $\pi_L(s_L, s^*_R|m, e) = \Pr(\mu \leq 1/2 + (m/4)\,(1+\beta\sigma)\,/\,(1+\beta))$, which is now bigger than $\pi_L(s^*|m, e)$. We now derive $\pi_L(s_L, s^*_R|m, m)$. A fraction β of group-l voters believe that $t_L = m$ and, with probability σ, that $t_R = e$. The remaining fraction believe that $t_L = t_R = m$. Hence, $i_{l,in} = 1/2 + m\sigma/4$ and $i_{l,out} = 1/2$. All group-r voters believe that $t_L = t_R = m$ and therefore $i_{r,in} = i_{r,out} = 1/2$. It follows that $\pi_L(s_L, s^*_R|m, m) = 1/2 = \pi_L(s^*|m, m)$. Using these two facts we have that deviation 2 is not profitable if and only if

$$\alpha \geq \sigma^* \left(1 - \frac{3m}{2}\right) \left[\pi_L(s_L, s^*_R|m, e) - \pi_L(s^*|m, e)\right].$$

Given that the right hand side of the above inequality, which is a function of α, vanishes as α approaches $\bar{\alpha}$, there exists $\alpha^* \in (\max[0, \underline{\alpha}(\beta)], \bar{\alpha})$ such that the inequality holds if $\alpha > \alpha^*$.