Generalized Systematic Risk

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July 2015

Online Appendix

Mean-Risk Preferences and Expected Utility

Background

One would wonder how the mean-risk preferences considered in Section III are related to the commonly assumed von Neumann-Morgenstern utility. It is widely known that a von Neumann-Morgenstern investor with a quadratic utility function only cares about the mean and the variance of his investments in the sense that he prefers a high expected wealth and a low variance. In this sense, the mean-risk preference is consistent with the von Neumann-Morgenstern utility when variance is used as the risk measure. Alternatively, when returns are distributed according to a two-parameter elliptical distribution (normal being a special case), mean-variance preferences can also be supported by expected utility. These instances, however, are quite restrictive. First, the quadratic utility is not very intuitive since it implies increasing absolute risk aversion. Second, elliptical distributions, being determined by the first two moments only, limit our ability to describe the dependence of risk on high distribution moments and other risk characteristics. Thus, in general, mean-variance preferences are not consistent with expect utility. The approach taken in this paper is much more general, allowing for a variety of risk measures. Whether a particular risk measure is consistent with expected utility depends on the actual choice of the risk measure. For example, risk measures that are simple linear combinations of raw moments up to the k^{th} degree can be represented by a k^{th} degree polynomial (Müller and Machina (1987)), generalizing the mean-variance result.

While in general the preferences defined in (6) cannot be supported by expected utility, they are often consistent with expected utility *locally*. The idea is based on Machina's (1982) "Local Utility Function." To facilitate this approach we first restrict attention to risk measures that depend on the distribution of the random variables only. Thus, we consider risk measures that are functions from the distribution of realizations to the reals rather than functions from the random variables themselves. Practically, this does not present a binding restriction since all the examples in this paper and all standard risk measures only rely on the distribution of realizations anyway. In this case the preferences in (6) can be written as

$$U(\boldsymbol{\zeta}) = V\left(\mathrm{E}\left(F_{\boldsymbol{\zeta}\cdot\widetilde{\mathbf{y}}}\right), R\left(F_{\boldsymbol{\zeta}\cdot\widetilde{\mathbf{y}}}\right)\right),$$

where $F_{\boldsymbol{\zeta}\cdot\widetilde{\mathbf{y}}}$ is the cumulative distribution of the random variable $\boldsymbol{\zeta}\cdot\widetilde{\mathbf{y}}$. When the random variable of interest is clear, we will omit it from the notation and write the utility as $U(F) = V(\mathrm{E}(F), R(F))$.

According to Machina (1982), if the realizations of all random variables are contained in some bounded and closed interval I and $U(F)$ is Fréchet differentiable with respect to the L^1 norm,[1] then for any two distributions F_1, F_2 on I there exists $u(\cdot; F_1)$ differentiable almost everywhere on I such that

(A19)

$$U(F_2) - U(F_1) = \int_I u(y; F_1)\, dF_2(y) - \int_I u(y; F_1)\, dF_1(y) + o\left(\|F_2 - F_1\|\right),$$

where $\|\cdot\|$ denotes the L^1 norm. That is, starting from a wealth distribution F_1, if an investor moves to another "close" distribution F_2, then he compares the utility from these two distributions as if he is maximizing his expected utility with a local utility function $u(\cdot; F_1)$.

The key to applying Machina's result is to find sufficient conditions on the risk measure which guarantee that $U(F)$ is Fréchet differentiable. This can be done in many ways. Next we provide one simple but effective approach which is sufficient to validate many popular risk measures as consistent with local expected utility.

Risk Measures as Functions of Moments

Let $\mu_k^F = \int y^k dF(y)$ be the k^{th} raw moment given distribution F, and $m_k^F = \int \left(y - \mu_1^F\right)^k dF(y)$ be the k^{th} central moment given distribution F. Consider risk measures which are a function of a finite number of (raw or central) moments. We denote such risk measures by $R\left(\mu_{j_1}^F, ..., \mu_{j_l}^F, m_{k_1}^F, ..., m_{k_n}^F\right)$. We assume that R is differentiable in all arguments. The utility function in (6) then takes the form

(A20)
$$U(F) = V\left(\mu_1^F, R\left(\mu_{j_1}^F, ..., \mu_{j_l}^F, m_{k_1}^F, ..., m_{k_n}^F\right)\right),$$

[1] Fréchet differentiability is an infinite dimensional version of differentiability. The idea here is that $U(F)$ changes smoothly with F, where changes in F are topologized using the L^1 norm. See Luenberger (1969, p. 171).

where V is differentiable in both mean and risk. This class of utility functions is quite general and it allows the risk measure to depend on a large number of high distribution moments. We then have the following proposition.

Proposition 1 *If $U(F)$ takes the form (A20) then for any two distributions F_1, F_2 on I there exists $u(\cdot; F_1)$ differentiable almost everywhere on I such that (A19) holds.*

Proof: We need to show that $U(F)$ is Fréchet differentiable. By the chain rule for Fréchet differentiability (Luenberger (1969, p. 176)), we know that if both μ_k^F and m_k^F are Fréchet differentiable for any k, then so is $U(\cdot)$. The Fréchet differentiability of μ_k^F is obvious, since

$$\mu_k^{F_2} - \mu_k^{F_1} = \int_I y^k dF_2(y) - \int_I y^k dF_1(y) = -k \int_I (F_2(y) - F_1(y)) y^{k-1} dy.$$

Now we show that m_k^F is Fréchet differentiable. We have

$$\begin{aligned} m_k^F &= \int \left(y - \mu_1^F\right)^k dF(y) \\ &= \int \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} y^i \left(\mu_1^F\right)^{k-i} dF(y) \\ &= \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \left(\mu_1^F\right)^{k-i} \int y^i dF(y) \\ &= \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \left(\mu_1^F\right)^{k-i} \mu_i^F, \end{aligned}$$

which is a differentiable function of the μ_i^F's. By the chain rule, it follows immediately that m_k^F is also Fréchet differentiable. This completes the proof. ■

Sufficient Conditions for Positive Prices

Here we provide a sufficient condition for the positivity of equilibrium prices following the approach of Nielsen (1992). Let $\boldsymbol{\zeta} \in \mathcal{R}^{n+1}$ be a bundle. Denote the gradient of investor j's utility function at $\boldsymbol{\zeta}$ by $\nabla \mathbf{U}^j(\boldsymbol{\zeta}) = \left(U_0^j(\boldsymbol{\zeta}), ..., U_n^j(\boldsymbol{\zeta})\right)$, where a subscript designates a partial derivative in the direction of the i^{th} asset. Also, let $\gamma^j(\boldsymbol{\zeta}) = -\left(V_2^j\left(\mathrm{E}(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}}), R(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}})\right)\right) / \left(V_1^j\left(\mathrm{E}(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}}), R(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}})\right)\right) > 0$ be the marginal rate of substitution of the expected payoff of the bundle for the risk of the bundle. This is the slope of investor j's indifference curve in the expected payoff-risk space. For brevity we often omit the arguments of this expression and use $\gamma^j(\boldsymbol{\zeta}) = -V_2^j/V_1^j$.

Proposition 2 *Assume that for each asset i there is some investor j such that $\mathrm{E}(\tilde{y}_i) > \gamma^j(\boldsymbol{\zeta}) R_i(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}})$ for all $\boldsymbol{\zeta}$. Then, prices of all assets are positive in all equilibria.*

Proof: At an equilibrium, all investors' gradients point in the direction of the price vector. So the price of asset i must be positive in any equilibrium if there is some investor j such that $U_i^j(\boldsymbol{\zeta}) > 0$ for all $\boldsymbol{\zeta}$. Recall that

$$U^j(\boldsymbol{\zeta}) = V^j(\mathrm{E}(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}}), R(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}})).$$

Thus,

$$\begin{aligned} U_i^j(\boldsymbol{\zeta}) &= V_1^j \mathrm{E}(\tilde{y}_i) + V_2^j R_i(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}}) \\ &= V_1^j [\mathrm{E}(\tilde{y}_i) - \gamma^j(\boldsymbol{\zeta}) R_i(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}})], \end{aligned}$$

where $R_i(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}})$ denotes the partial derivative of $R(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}})$ with respect to ζ_i.

Since $V_1^j > 0$, $U_i^j(\boldsymbol{\zeta}) > 0$ corresponds to

$$\mathrm{E}(\tilde{y}_i) - \gamma^j(\boldsymbol{\zeta}) R_i(\boldsymbol{\zeta} \cdot \widetilde{\mathbf{y}}) > 0,$$

as required. ■

Note that $\gamma^j(\cdot)$ can serve as a measure of risk aversion for investor j. We can thus interpret this proposition as follows. If each asset's expected return is sufficiently high relative to some investor's risk aversion and the marginal contribution of the asset to total risk, then this asset will always be desirable by some investor, and so, its price will be positive in any equilibrium.