Asymmetric Contests with Head Starts and Non-Monotonic Costs

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Online Appendix

1 Simple Examples

To gain some intuition for how the algorithm works, consider the contest with two players whose cost functions are depicted in Figure 1. The common value of the prize is 1, player 1 has zero costs up to $a_1 < 1$ and marginal costs 1 starting from a_1, and player 2 has marginal costs 1 starting from 0.[1] This is a special case of the all-pay auctions with head starts considered in Section III of the paper.

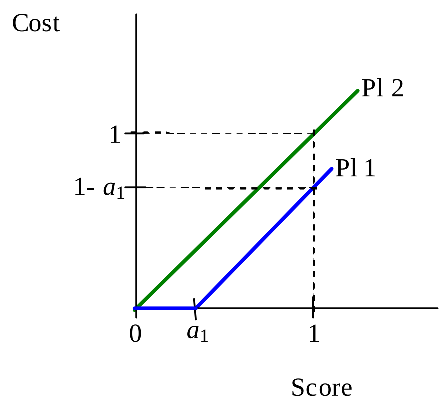

Figure 1: Players' costs in a two-player contest with valuations and constant marginal costs of 1 and a head start for player 1

The contest's threshold is 1 (equal to player 2's reach), player 1's power is a_1, and player 2's power is 0. Because player 2's costs are increasing at 0, Stage 1 specifies that $x_0 = 0$. By Lemma 1, $G_2(0) = a_1 < 1$, and Stage 2 shows that $\mathcal{CP}(0) = \{1, 2\}$. Because player 1's costs are constant at 0, the algorithm proceeds to Stage 3, which specifies that $\mathcal{A}^+(0) = \{1\}$. Therefore, the first checkpoint above 0 is $\bar{x} = a_1$, and no player's CDF increases in $(0, a_1)$. From (4) we have that $G_1(a_1) = a_1$. The algorithm proceeds to Stage 2, and we have that $\mathcal{CP}(a_1) = \{1, 2\}$. Because both players' costs are increasing at a_1, the algorithm proceeds to Stage 4, and $\mathcal{A}^+(a_1) = \{1, 2\}$. From (5) we have that $G_1(y) = 1 - q_1(y) q_2(y) / q_1(y) = 1 - q_2(y) = y$ and $G_2(y) = a_1 + y - a_1 = y$ for scores y in $[a_1, \bar{x}]$, where $\bar{x} = 1$ is the score at which players' CDFs reach 1. We then have $x^L = 1$, and Stage 5 specifies that both players' CDFs equal 1 starting from 1. Players's atoms

[1] If $a_1 \geq 1$, then player 2 has an atom of size 1 at 0 and the algorithm proceeds to Stage 5, which specifies that player 1 has an atom of size 1 at a_1.

and densities in the equilibrium are depicted in Figure 2.

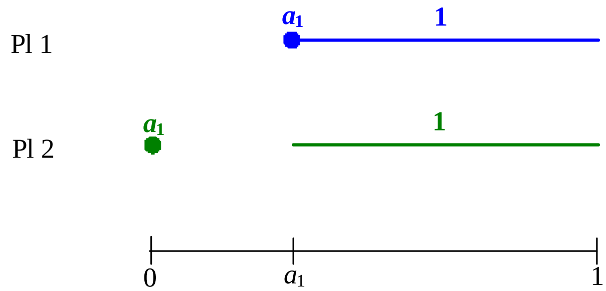

Figure 2: Players' atoms and densities in the equilibrium of the contest depicted in Figure 1

It may also be the case that a player with a head start is disadvantaged in terms of his marginal cost above his head start. To see what happens in this case, consider the two-player contest depicted in Figure 3, in which the marginal player has a head start. The common value of the prize is 1, player 1 has marginal costs $1 - a_1$ starting from 0, and player 2 has zero costs up to $a_1 < 1$ and marginal costs $1/(1 - a_1)$ starting from a_1.

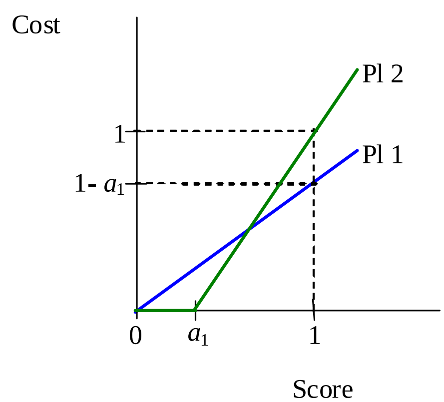

Figure 3: Players' costs in a two-player contest with differing marginal costs and a head start for player 2

The threshold is 1, player 1's power is a_1, and player 2's power is 0. Stage 1 specifies that $x_0 = a_1$, the highest score for which player 2's costs are 0. By Lemma 1, $G_2(a_1) = (1 - a_1) a_1 + a_1 = 2a_1 - a_1^2 < 1$, and Stage 2 shows that $\mathcal{CP}(a_1) = \{1, 2\}$. Because both players' costs are increasing at a_1, the algorithm proceeds to Stage 4 and we have that $\mathcal{A}^+(a_1) = \{1, 2\}$. From (5) we have that $G_1(y) = (y - a_1)/(1 - a_1)$ and $G_2(y) = a_1 + y(1 - a_1)$ for scores y in $[a_1, \bar{x}]$, where $\bar{x} = 1$ is the score at which players' CDFs reach 1. We then have $x^L = 1$, and Stage 5 specifies that both players' CDFs equal 1 starting from 1. Players's atoms and densities in the equilibrium are depicted in Figure 4.

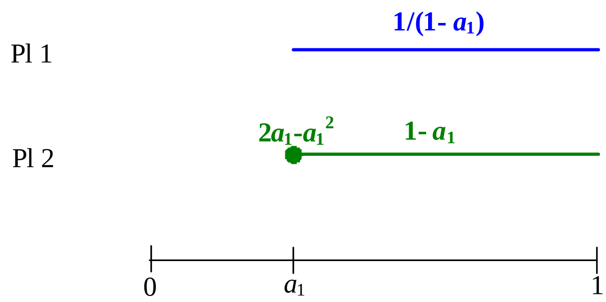

Figure 4: Players' atoms and densities in the equilibrium of the contest depicted in Figure 3

To see the importance of Stage 5, consider the two-player contest depicted in Figure 5. The common value of the prize is 1, player 1 has marginal costs 1 up to a_1, marginal costs 0 in $[a_1, 1]$, and marginal costs 1 above 1, and player 2 has marginal costs 1 starting from 0.

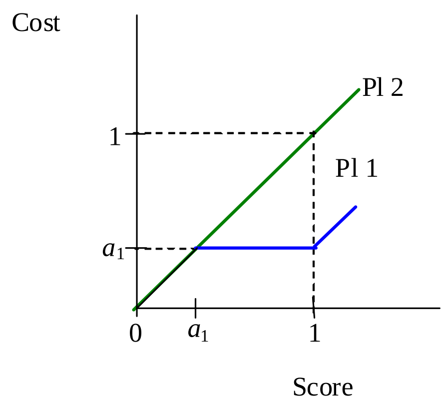

Figure 5: Players' costs in a two-player contest in which player 1 has an intermediate region with marginal costs 0

The threshold is 1, player 1's power is $1 - a_1$, and player 2's power is 0. Because player 2's costs are increasing at 0, Stage 1 specifies that $x_0 = 0$. By Lemma 1, $G_2(0) = 1 - a_1 < 1$, and Stage 2 shows that $\mathcal{CP}(0) = \{1, 2\}$. Because both players' costs are increasing at 0, the algorithm proceeds to Stage 4 and we have that $\mathcal{A}^+(a_1) = \{1, 2\}$. From (5) we have that $G_1(y) = y$ and $G_2(y) = 1 - a_1 + y$ for scores y in $[0, \bar{x}]$, where $\bar{x} = a_1$ is the first score at which player 1's costs are constant. Because player 2's CDF reaches 1 at a_1, we have that $x^L = a_1$. The algorithm proceeds to Stage 5, which specifies that player 1's remaining probability mass is expended at 1, the highest score whose cost for player 1 is equal to his cost of choosing a_1. Both players' CDFs equal 1 for scores higher than 1. Players's atoms and densities in the equilibrium are depicted in Figure 6.

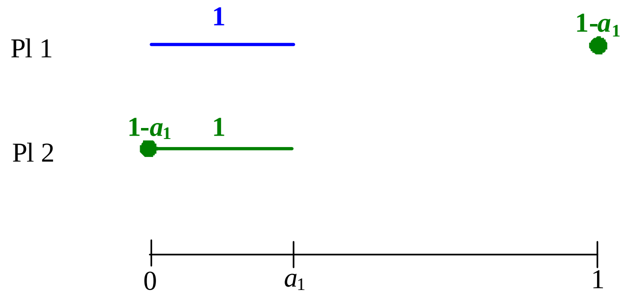

Figure 6: Players' atoms and densities in the equilibrium of the contest depicted in Figure 3

2 Derivation of the Equilibrium for the Three-Player, Two-Prize All-Pay Auction with Head Starts

Players' powers are $w_1 = a_1 - a_3$, $w_2 = a_2 - a_3$, and $w_3 = 0$. Stage 1 specifies that $x_0 = a_3$ and $G_3(a_3) = (a_2 - a_3)/V$. Stage 2 shows that $\mathcal{CP}(a_3) = \{2, 3\}$. Because player 2's costs are constant at a_3, the algorithm proceeds to Stage 3, which gives $\mathcal{A}^+(a_3) = \{2\}$. Therefore, the first checkpoint above a_3 is a_2, and no player's CDF increases in (a_3, a_2). From (3) we have

$$G_2(a_2) = \min\left\{1 - \frac{1 - \frac{w_1 + c_1(a_2)}{V}}{1 - G_3(a_3)}, 1 - \frac{1 - \frac{w_3 + c_3(a_2)}{V}}{1 - G_1(a_3)}\right\}$$

$$= \min\left\{\frac{a_1 - a_2}{a_3 + V - a_2}, \frac{a_2 - a_3}{V}\right\} = \begin{cases} \frac{a_2 - a_3}{V} & \text{if } a_1 \geq 2a_2 - a_3 - \frac{(a_2 - a_3)^2}{V} \\ \frac{a_1 - a_2}{a_3 + V - a_2} & \text{if } a_1 < 2a_2 - a_3 - \frac{(a_2 - a_3)^2}{V} \end{cases}.$$

This implies that

$$\mathcal{CP}(a_2) = \begin{cases} \{2, 3\} & \text{if } a_1 > 2a_2 - a_3 - \frac{(a_2 - a_3)^2}{V} \\ \{1, 2, 3\} & \text{if } a_1 = 2a_2 - a_3 - \frac{(a_2 - a_3)^2}{V} \\ \{1, 2\} & \text{if } a_1 < 2a_2 - a_3 - \frac{(a_2 - a_3)^2}{V} \end{cases}.$$

Case 1, $a_1 > 2a_2 - a_3 - (a_2 - a_3)^2/V$: Because the cost functions of players 2 and 3 are increasing at a_2, the algorithm proceeds to Stage 4. Because $\mathcal{A}^+(a_2) = \mathcal{CP}(a_2) = \{2, 3\}$, from (5) we have that $G_2(y) = 1 - q_3(y) = (y - a_3)/V$ and $G_3(y) = 1 - q_2(y) = (y - a_3)/V$ for scores y in $[a_2, \bar{x}]$, where the checkpoint $\bar{x}$ is the first score at which player 1 obtains his power:

$$\left(1 - \left(1 - \frac{\bar{x} - a_3}{V}\right)^2\right) V = a_1 - a_3 \Rightarrow \bar{x} = a_3 + V\left(1 - \sqrt{1 - \frac{a_1 - a_3}{V}}\right) > a_2.$$[2]

[2] That the inequality holds is true because it is equivalent to the inequality $a_1 > 2a_2 - a_3 - \frac{(a_2 - a_3)^2}{V}$ (an equivalence between the reverse inequalities also holds).

Proceeding to Stage 2, we have that $\mathcal{CP}(\bar{x}) = \{1,2,3\}$, and because the cost function of player 1 is constant at $\bar{x}$, the algorithm proceeds to Stage 3, which gives $\mathcal{A}^+(\bar{x}) = \{1\}$. Therefore, the next checkpoint above $a_3 + V\left(1 - \sqrt{1-(a_1-a_3)/V}\right)$ is a_1 and no player's CDF increases in $\left(a_3 + V\left(1 - \sqrt{1-(a_1-a_3)/V}\right), a_1\right)$. From (3) we have

$$G_1(a_1) = \min\left\{1 - \frac{1 - \frac{w_2+c_2(a_1)}{V}}{1 - G_3\left(a_3 + V\left(1 - \sqrt{1 - \frac{a_1-a_3}{V}}\right)\right)}, 1 - \frac{1 - \frac{w_3+c_3(a_1)}{V}}{1 - G_2\left(a_3 + V\left(1 - \sqrt{1 - \frac{a_1-a_3}{V}}\right)\right)}\right\}$$

$$= \min\left\{1 - \sqrt{1 - \frac{a_1-a_3}{V}}, 1 - \sqrt{1 - \frac{a_1-a_3}{V}}\right\} = 1 - \sqrt{1 - \frac{a_1-a_3}{V}}.$$

Proceeding to Stage 2, we have that $\mathcal{CP}(a_1) = \{1,2,3\}$, and because all players' costs are increasing at a_1 the algorithm proceeds to Stage 4. Because $\mathcal{A}^+(a_1) = \mathcal{CP}(a_1) = \{1,2,3\}$, from (5) we have

$$G_i(y) = 1 - \frac{\Pi_{j\in\{1,2,3\}} q_j(y)^{\frac{1}{|\{1,2,3\}|-1}}}{q_i(y)} = 1 - \frac{\left(\frac{V+a_3-y}{V}\right)^{\frac{3}{2}}}{\frac{V+a_3-y}{V}} = 1 - \sqrt{1 - \frac{y-a_3}{V}} \tag{1}$$

for every player i and score y in $[a_1, \bar{x}]$, where the checkpoint $\bar{x} = a_3 + V$ is the score at which players' CDFs reach 1.

Case 2, $a_1 = 2a_2 - a_3 - (a_2-a_3)^2/V$: Because the cost function of player 1 is constant at a_2, the algorithm proceeds to stage 3, which gives $\mathcal{A}^+(a_2) = \{1\}$. Therefore, the next checkpoint above a_2 is a_1 and no player's CDF increases in (a_2, a_1). The equilibrium starting from a_1 is as in Case 1: $G_i(y) = 1 - \sqrt{1-(y-a_3)/V}$ for every player i and score y in $[a_1, a_3+V]$. All players' CDF reach 1 at $a_3 + V$.

Case 3, $a_1 < 2a_2 - a_3 - (a_2-a_3)^2/V$: Because the cost function of player 1 is constant at a_2, the algorithm proceeds to stage 3, which gives $\mathcal{A}^+(a_2) = \{1\}$. Therefore, the next checkpoint above a_2 is a_1, and no player's CDF increases in (a_2, a_1). From (3) we have

$$G_1(a_1) = \min\left\{1 - \frac{1 - \frac{w_2+c_2(a_1)}{V}}{1 - G_3(a_2)}, 1 - \frac{1 - \frac{w_3+c_3(a_1)}{V}}{1 - G_2(a_2)}\right\} = \min\left\{\frac{a_1-a_2}{a_3+V-a_2}, \frac{a_2-a_3}{V}\right\} = \frac{a_1-a_2}{a_3+V-a_2}.$$

Proceeding to Stage 2, we have that $\mathcal{CP}(a_1) = \{1,2\}$, and because both players' costs are increasing at a_1 the algorithm proceeds to Stage 4. Because $\mathcal{A}^+(a_1) = \mathcal{CP}(a_1) = \{1,2\}$, from (5) we have

$$G_1(y) = 1 - \frac{q_1(y)\,q_2(y)}{q_1(y)(1 - G_3(a_1))} = \frac{y-a_2}{a_3+V-a_2} = 1 - \frac{q_1(y)\,q_2(y)}{q_2(y)(1-G_3(a_1))} = G_2(y)$$

for scores y in $[a_1, \bar{x}]$, where the checkpoint $\bar{x}$ is the first score at which player 3 obtains his power:

$$\left(1 - \left(1 - \frac{\bar{x}-a_2}{a_3+V-a_2}\right)^2\right)V - (\bar{x}-a_3) = 0 \Rightarrow \bar{x} = 2a_2 - a_3 - \frac{(a_2-a_3)^2}{V} > a_1.$$

Proceeding to Stage 2, we have that $\mathcal{CP}(\bar{x}) = \{1,2,3\}$, and because all players' costs are increasing at $\bar{x}$ the algorithm proceeds to Stage 4. Because $\mathcal{A}^+(\bar{x}) = \mathcal{CP}(\bar{x}) = \{1,2,3\}$, (1) tells us that $G_i(y) = 1 - \sqrt{1-(y-a_3)/V}$ for every player i and score y in $[\bar{x}, a_3+V]$. All players' CDFs reach 1 at $a_3 + V$.

3 Contest Design

The payoff and equilibrium characterizations can also be used for contest design. As an example, consider an $(m+1)$-player all-pay auction with heads starts and m prizes of value $V > a_1$, and two types of intervention by the contest administrator: handicaps and subsidies. A handicap increases the handicapped player's score without affecting his output and at no direct cost to the administrator. A subsidy increases the player's output (and therefore his score) by allocating some of the prize money to defray the player's expenditures. Because of the linearity of players' costs (above their head starts), the payoff result is sufficient to determine the optimal handicaps and subsidies when the goal is to maximize aggregate expected expenditures or output.[3]

Because every player i's power is $a_i - a_{m+1}$, the aggregate expected expenditures are

$$mV - \sum_{i=1}^{m} a_i + m a_{m+1}.$$

If handicaps $h_1, \ldots, h_{m+1}$ are administered, the resulting game is strategically equivalent to an all-pay auction in which every player i's head start is $a_i + h_i$. This implies that handicaps $h_i = a_1 - a_i$ lead to aggregate expected expenditures of

$$mV - \sum_{i=1}^{m}(a_i + h_i) + m(a_{m+1} + h_{m+1}) = mV - \sum_{i=1}^{m}(a_i + a_1 - a_i) + m(a_{m+1} + a_1 - a_{m+1}) = mV.^4$$

These are the highest possible aggregate expected expenditures, because aggregate expected expenditures are bounded above by the aggregate value of the prizes. Note that a player's output equals his head start plus his expenditures (the handicap is not added to the output), so maximizing aggregate output is equivalent to maximizing aggregate expected expenditures. Therefore, the maximal aggregate expected output of $mV + \sum_{i=1}^{m+1} a_i$ is achieved by setting $h_i = a_1 - a_i$. This analysis also implies that adding additional players cannot increase the aggregate expenditures or output.

If subsidies are administered, then a subsidy of b_i increases player i's output by b_i and reduces the amount of prize money by b_i. To maximize aggregate expected output, set $b_i = a_1 - a_i$ (just

[3] The output is the "real" score, excluding handicaps but including subsidies. This could correspond, for example, to the quality or quantity of what is produced in the course of the competition.

[4] Although Condition M3 is violated when all players are identical, Corollary 3 in Siegel (2009) shows that in this case every player's equilibrium payoff is 0.

like the handicaps above). This gives the maximal aggregate expected output of

$$\underbrace{\underbrace{mV - \sum_{i=1}^{m+1} b_i}_{\text{prize value}} - \underbrace{\left(\sum_{m=1}^{m} (a_i + b_i) - m\,(a_{m+1} + b_{m+1})\right)}_{\text{expected payoffs}}}_{\text{expected expenditures}} + \underbrace{\sum_{i=1}^{m+1} (a_i + b_i)}_{\text{subsidies and head starts}}$$

$$= mV - \textstyle\sum_{i=2}^{m+1} (a_1 - a_i) + (m+1)\,a_1 = mV + \sum_{i=1}^{m+1} a_i,$$

just like with optimal handicaps.

A trade-off is introduced when subsidies are chosen to maximize aggregate expected expenditures. This is because increasing the marginal player's output enhances competition, which increases expenditures, but also decreases the aggregate prize value, which decreases competition and lowers expenditures. As long as $a_i + b_i \geq a_j + b_j$ for $i < j$, aggregate expenditures are

$$mV - \sum_{i=1}^{m+1} b_i - \left(\sum_{m=1}^{m} (a_i + b_i) - m\,(a_{m+1} + b_{m+1})\right),$$

so subsidizing a player who is not the marginal player lowers expenditures both because it increases the player's payoff and because it lowers the prize value. Subsidizing the marginal player increases expenditures by $(m-1)\,b_{m+1}$, so it is optimal to give the marginal player a subsidy of $a_m - a_{m+1}$. At this point, any additional subsidy to player $m+1$ requires that the same additional subsidy be give to player m (otherwise player m would become the marginal player and expenditures would decrease, as discussed above). Therefore, an additional subsidy of z increases expenditures by $(m-3)\,z$. If $m \geq 3$ then it is optimal to set an additional subsidy of $z = a_{m-1} - a_m$. At this point, any additional subsidy must be given to player $m-1$ as well. Continuing in this way, we see that the optimal subsidy equates the head starts of players $k, k+1, \ldots, m+1$ with that of player $k-1$, where $k \geq 2$ is the lowest integer that satisfies

$$m \geq (m+1-k) + 1 + m - k + 1 \iff m \geq 2m - 2k + 3 \iff k \geq \frac{m+3}{2}.$$[5]

Design questions for which knowledge of players' equilibrium payoffs is insufficient may be addressed by applying the equilibrium construction algorithm. I illustrate this by briefly considering some design questions in a three-player all-pay auction with head starts and two prizes of value V. Players' equilibrium strategies are described in Section 4.1 in the text. Suppose that we are interested in increasing player 1's expenditures (and output) by subsidizing or handicapping player 2. If $a_1 > 2a_2 - a_3 - (a_2 - a_3)^2 / V$, then Figure 4 in the text describes the equilibrium

[5] It can be shown that in an $(m+1)$-player all-pay auction with head starts players' payoffs equal their power and there is a unique equilibrium, given by the algorithm, even when Condition M3 does not hold. This continues to be true if additional players with head starts lower than a_{m+1} are added. These additional players do not participate.

as long as the subsidy/handicap is not too large. Considering player 1's equilibrium CDF in this case, it is immediate that a handicap for player 2 has no effect, and a subsidy for player 2 reduces player 1's expenditures because it lowers the prizes' value. If Figure 5 in the text describes the equilibrium, then it can be shown that a handicap for player 2 of up to $a_1 - a_2$ monotonically increases player 1's expenditures. This implies that a handicap of $a_1 - a_2$ is optimal. It can be shown, however, that if a_1 is small, then any subsidy to player 2 reduces player 1's expenditures.[6] Another object of possible interest is the first-order statistic of players' scores (this would correspond to the quality of the best innovation in an R&D setting). If Figure 4 in the text describes the equilibrium, it can be shown that the first-order statistic is convex in a_1. This implies that the optimal subsidy to player 1 is either 0 or the maximal subsidy (so that $a_1 + b_1 = V - \frac{b_1}{2}$). The former is optimal if a_1 is small, the latter is optimal if a_1 is large.

[6] It can also be shown that if Figure 4 in the text describes the equilibrium, then a subsidy to player 1 increases his output: the decrease in expenditures due to the lower prize value is smaller than the increase in score due to the subsidy.