Delay and Deadlines: Freeriding and Information Revelation in Partnerships

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Online Appendix

B Additional Propositions and Proofs

B.1 Proofs of Corollaries

Proof of Corollary 1.
This follows immediately from

$$\exp\left(-\lambda e_{\max} X\right) = \bar{\phi} \text{ and } \lambda\left[\bar{\phi}\alpha_1 + \left(1-\bar{\phi}\right)\alpha_2\right] = c.$$

If either c decreases, α_1 increases or α_2 increases, $\bar{\phi}$ decreases and thus X increases. If $e_{\max}$ increases, clearly X decreases. Finally, using $X = -\frac{1}{\lambda e_{\max}} \log \frac{\frac{c}{\lambda}-\alpha_2}{\alpha_1-\alpha_2}$, we find that $\frac{dX}{d\lambda} > 0$ if and only if $-\log\bar{\phi} < \frac{c}{\bar{\phi}\lambda} = e_{\max}\alpha_2$.■

Proof of Corollary 2.
For $T > X$, an equilibrium with private information exists in which $e^{*,priv}(t) = 0$ for $t < T - X$ and $e^{*,priv}(t) = e_{\max}$ for $t \geq T - X$. In the unique equilibrium with public information $e^{*,pub}(t) = e_{\max}$ if and only if $t \geq T - \Delta$. We show that $X > \Delta$ by contradiction. By definition, X is the deadline T solving

$$V_T^{S,priv}(0) - V_T^{U,priv}(0) = \frac{c}{\lambda} \Leftrightarrow \alpha_1\phi^*(T) + \alpha_2\left(1-\phi^*(T)\right) = \frac{c}{\lambda},$$

where $\phi^*(T) = \exp\left(-\lambda e_{\max} T\right)$. Also, by definition, Δ is the deadline T solving

$$V_T^{S,pub}(0) - V_T^{U,pub}(0) = \frac{c}{\lambda} \Leftrightarrow \alpha_1\phi^*(T)^2 + \left(\frac{c}{2\lambda} + \frac{\delta}{2\lambda e_{\max}}\right)\left(1-\phi^*(T)^2\right) = \frac{c}{\lambda},$$

where we use $\exp\left(-2\lambda e_{\max} T\right) = \phi^*(T)^2$. Since $\phi^*(T)^2 \leq \phi^*(T)$, a necessary condition for $X < \Delta$ is

$$\alpha_2 < \frac{c}{2\lambda} + \frac{\delta}{2\lambda e_{\max}}, \tag{1}$$

which implies that exerting effort to produce a second breakthrough is socially inefficient. We now show that when this inequality holds, welfare under public information exceeds the welfare under private information for short deadlines,

$$V_T^{U,pub}(0) - V_T^{U,priv}(0) > 0 \text{ for } T \leq \min\{X, \Delta\}. \tag{2}$$

However, from Proposition 5, we have

$$V_X^{U,priv}(0) \geq V_\Delta^{U,pub}(0) = \max_T V_T^{U,pub}(0) \geq V_X^{U,pub}(0).$$

Hence, this implies that $X > \Delta$, which is a contradiction. To establish the inequality (2), we use that for $T \leq \Delta$,

$$V_T^{U,pub}(0) = V_0 + \left(\alpha_1 - \frac{c}{2\lambda} - \frac{\delta}{2\lambda e_{\max}}\right)\left(1-\exp\left(-2\lambda e_{\max} T\right)\right),$$

and for $T < X$,

$$V_T^{U,priv}(0) = V_0 + \alpha_1\left(1-\phi^*(T)^2\right) + \alpha_2\left(1-\phi^*(T)\right)^2 - \frac{c}{\lambda}\left(1-\phi^*(T)\right) - \delta T.$$

Using $\exp(-2\lambda e_{\max} T) = \phi^*(T)^2$, we find that higher welfare is achieved in the public information case if

$$\alpha_2 (1 - \phi^*(T))^2 - \frac{c}{\lambda}(1 - \phi^*(T)) - \delta T \leq -\frac{c}{2\lambda}\left(1 - \phi^*(T)^2\right) - \frac{\delta}{2\lambda e_{\max}}\left(1 - \phi^*(T)^2\right).$$

Rearranging, we find

$$\left(\alpha_2 - \frac{c}{2\lambda} - \frac{\delta}{2\lambda e_{\max}}\right)(1 - \phi^*(T))^2 \leq \delta\left(T - \frac{1 - \phi^*(T)}{\lambda e_{\max}}\right).$$

The term $\frac{1-\phi^*(T)}{\lambda e_{\max}}$ corresponds to the expected duration of a game with maximum length T when the project is implemented at the rate $\lambda e_{\max}$ and is thus smaller than T. Hence, the right-hand side has a positive sign. Moreover, from inequality (1), we know that the left-hand side has a negative sign. This establishes the inequality.■

Proof of Corollary 3.

Knowing that $T < \Delta$ implies $T < X$ by Corollary 2, this follows immediately from the second part of the proof of that Corollary.■

Proof of Corollary 4.

The condition $(2\lambda\alpha_2 - c)\, e_{\max} < \delta$ implies that it is socially efficient for the team to produce only one breakthrough. The efficient level of the value of the project is thus bounded above by $V_0 + \alpha_1$. However, for $T \in [X, Y]$, the expected value of the project equals $V_0 + \left(1 - \bar{\phi}\right)^2(\alpha_1 + \alpha_2) + 2\left(1 - \bar{\phi}\right)\bar{\phi}\,(V_0 + \alpha_1)$. When $\frac{\alpha_2}{\alpha_1} > \frac{\bar{\phi}^2}{\left(1-\bar{\phi}\right)^2}$, this exceeds $V_0 + \alpha_1$.■

B.2 Large Effort Incentives ($\bar{\phi}_d \geq \bar{\phi}$)

In this section, we describe the case where effort incentives are large ($\bar{\phi}_d \geq \bar{\phi}$, see Section 6.1) in more detail. We also state the equilibrium strategies and beliefs formally and provide a proof.

We consider deadlines of different length T of which there are four distinct cases. We define two thresholds X_d and Y_d, similar to X and Y, and an additional threshold Z. The threshold X_d denotes the length of time necessary such that an agent exerting maximum effort is successful with probability $1 - \bar{\phi}_d$. Thus, the threshold solves

$$\exp(-\lambda e_{\max} X_d) = \bar{\phi}_d.$$

Similarly, the threshold Y_d denotes the length of time that an agent would be willing to delay implementation in exchange for the additional benefit of a second breakthrough with probability $1 - \bar{\phi}_d$. Thus, the threshold solves

$$\left(1 - \bar{\phi}_d\right)\alpha_2 = \delta Y_d.$$

The characterization of the equilibrium is very similar as before, with the exception of a final stage which lasts up to Z for games with length exceeding X_d. Once the length of the game exceeds X_d and unsuccessful agents have exerted maximum effort until $t = X_d$, successful players will implement projects at a rate $d^*(t)$ such that $\left(1 - \bar{\phi}_d\right) d^*(t) = \lambda e_{\max}$ keeping the belief constant at $\bar{\phi}_d$. The threshold Z is the maximum length of this mixing stage which maintains maximum incentives to exert effort throughout the game,

$$Z = -\frac{1}{2\lambda e_{\max}} \log\left(-\frac{\left(1 - \bar{\phi}_d\right)\alpha_2 - \frac{c e_{\max} - \delta}{\lambda e_{\max}}}{\bar{\phi}_d 2\alpha_1 + \left(1 - \bar{\phi}_d\right)\alpha_2 - \frac{c e_{\max} + \delta}{\lambda e_{\max}}}\right).$$

Proposition 8. *If $\bar{\phi}_d \geq \bar{\phi}$, then the equilibrium strategies and beliefs are as follows:*
i) If $T \leq X_d$, any successful player chooses not to implement, $d^*(t) = 0$, for all t while any unsuccessful player chooses to exert maximum effort, $e^*(t) = e_{\max}$, for all t. The agents' beliefs evolve according to $\phi^*(t) = \exp(-\lambda e_{\max} t)$.
ii) If $X_d < T \leq X_d + Z$, any successful player chooses not to implement for $t < X_d$ and decides to implement at the mixing rate $d^*(t) = \frac{(\lambda e_{\max})^2 \alpha_2}{\lambda e_{\max}\alpha_2 - \delta}$ for $t \geq X_d$. Any unsuccessful player chooses to exert maximum effort, $e^*(t) = e_{\max}$, for all t. The agents' beliefs evolve according to $\phi^*(t) = \exp(-\lambda e_{\max} t)$ for $t \leq X_d$ and $\phi^*(t) = \bar{\phi}_d$ for $t > X_d$.

iii) If $X_d + Z < T \leq Y_d + Z$, any successful agent chooses not to implement for $t < t_d \equiv T - Z$, and decides to implement at the mixing rate $d^*(t) = \frac{(\lambda e_{\max})^2 \alpha_2}{\lambda e_{\max} \alpha_2 - \delta}$ for $t \geq t_d$. Any unsuccessful player chooses to exert effort $e^*(t)$ for $0 \leq t < t_d$ which is not uniquely determined, but the effort choice must satisfy the following conditions:

$$\phi^*(t) = \exp\left(-\lambda \int_0^t e^*(s)\, ds\right) \leq \frac{\delta}{\alpha_2}[t - t_0] \ \text{for } t \in [t_0, t_d] \tag{3}$$

and

$$\phi(t_d) = \exp\left(-\lambda \int_{t_0}^{t_d} e^*(s)\, ds\right) = \frac{\delta}{\lambda \alpha_2}, \tag{4}$$

for $t_0 = 0$. For $t \geq t_d$ the unsuccessful agent exerts maximal effort $e^*(t) = e_{\max}$. The agents' beliefs evolve according to $\phi^*(t) = \exp\left(-\lambda \int_0^t e^*(s)\, ds\right)$ for $0 \leq t \leq t_d$ and $\phi^*(t) = \bar{\phi}_d$ for $t > t_d$.
iv) If $T > Y_d + Z$, any successful player chooses to implement immediately for $t < t_d - Y_d$, not to implement, $d^*(t) = 0$ for $t_d - Y_d \leq t < t_d$, and to implement at the mixing rate $d^*(t) = \frac{(\lambda e_{\max})^2 \alpha_2}{\lambda e_{\max} \alpha_2 - \delta}$ for $t \geq t_d$. Any unsuccessful agent chooses to exert effort $e^*(t) = \frac{\delta}{c}$ for $t < t_d - Y_d$, and to exert effort $e^*(t)$ for $t_d - Y_d \leq t < t_d$ which is not uniquely determined but must satisfy the following conditions (3) and (7) for $t_0 = t_d - Y_d$, and to exert maximal effort $e^*(t) = e_{\max}$ for $t \geq t_d$.

Proof. **i) Case 1:** $T \leq X_d$

The proof is here exactly the same as for Case 1 in Proposition 1, since $\bar{\phi}_d = \exp(-\lambda e_{\max} X_d) > \bar{\phi}$ in this case with large effort incentives.

ii) **Case 2:** $X_d < T \leq X_d + Z$

We again start by writing out the implied continuation values of successful and unsuccessful agents on the equilibrium path for the proposed equilibrium strategies.

At the deadline, the continuation values equal

$$\begin{aligned} V^S(T) &= V_0 + \alpha_1 + \left(1 - \bar{\phi}_d\right) \alpha_2, \\ V^U(T) &= V_0 + \left(1 - \bar{\phi}_d\right) \alpha_1. \end{aligned}$$

For $t \in [X_d, T]$, the continuation values equal

$$V^S(t) = V^S(T)$$

and

$$V^U(t) = \int_t^T \left(V_0 + \alpha_1 + \frac{\left(1 - \bar{\phi}_d\right)\alpha_2}{2} - (ce_{\max} + \delta)(s - t)\right) 2\lambda e^{-2\lambda e_{\max}(s-t)} ds + e^{-2\lambda e_{\max}(T-t)} \left(V^U(T) - (ce_{\max} + \delta)(T - t)\right).$$

Note that the 2 in the density function comes from the probability of one of two events occuring, "agent successfully produces a breakthrough" or "the other agent decides to implement". In this time interval, the rate of breakthrough production is the same as the rate at which the other agent is deciding to implement the project. The payoff contains the term $\left(1 - \bar{\phi}_d\right)\alpha_2/2$ because with 1/2 probability the agent will produce a breakthrough before the other agent decides to implement the project in which case the payoff is increased by $\left(1 - \bar{\phi}_d\right)\alpha_2$. The continuation value simplifies further to

$$V^U(t) = V_0 + \alpha_1 + \frac{\left(1 - \bar{\phi}_d\right)\alpha_2}{2} - (ce_{\max} + \delta)\frac{1}{2\lambda e_{\max}} - e^{-2\lambda e_{\max}(T-t)} \left[\bar{\phi}_d \alpha_1 + \frac{\left(1 - \bar{\phi}_d\right)\alpha_2}{2} - (ce_{\max} + \delta)\frac{1}{2\lambda e_{\max}}\right].$$

The difference between being successful and unsuccessful is given by

$$V^S(t) - V^U(t) = \frac{\left(1-\bar{\phi}_d\right)\alpha_2}{2} + (ce_{\max}+\delta)\frac{1}{2\lambda e_{\max}} + e^{-2\lambda e_{\max}(T-t)}\left[\bar{\phi}_d\alpha_1 + \frac{\left(1-\bar{\phi}_d\right)\alpha_2}{2} - (ce_{\max}+\delta)\frac{1}{2\lambda e_{\max}}\right].$$

We define Z such that this difference equals exactly $\frac{c}{\lambda}$ when $t = T - Z$. Notice that $\frac{d\left(V^S(t)-V^U(t)\right)}{dt}|_{t=T-Z} > 0$, since $\frac{dV^S(t)}{dt}|_{t=T-Z} = 0$ and $\frac{dV^U(t)}{dt}|_{t=T-Z} < 0$.

For $t \in [0, X_d]$, the continuation values equal

$$\begin{aligned} V^S(t) &= V_0 + \alpha_1 + \left(1-\bar{\phi}_d\right)\alpha_2 - \delta(X_d - t), \\ V^U(t) &= \left[1-\exp\left(-\lambda e_{\max}(X_d-t)\right)\right]\left(V^S(X_d) - \frac{c}{\lambda}\right) + \exp\left(-\lambda e_{\max}(X_d-t)\right)V^U(X_d) - \delta\left(\hat{t}-t\right). \end{aligned}$$

The difference between the two equals

$$V^S(t) - V^U(t) = \frac{c}{\lambda} + e^{-\lambda(X_d-t)}\left(V^S(X_d) - V^U(X_d) - \frac{c}{\lambda}\right).$$

Strategy of successful player: Check first that the successful individual's implementation decision strategy $d^*(t) = 0$ is optimal for $t \in [0, X_d]$ by verifying whether $V^S(t) \geq V_0 + \alpha_1 + (1-\phi^*(t))\alpha_2$, where the right hand side equals the payoff from immediate implementation. Rearranging , we obtain

$$\phi^*(t) - \bar{\phi}_d \geq \frac{\delta}{\alpha_2}(X_d - t),$$

which holds with equality for $t = X_d$. Furthermore, the derivative of the LHS is strictly less than the RHS, $-\lambda e_{\max}\phi^*(t) < -\frac{\delta}{\alpha_2}$, for $t < X_d$. Hence, the relation holds for $t < X_d$. Next, we check that the successful individual is indifferent about a decision now versus delaying a decision any amount Δt into the future for $t \in (X_d, T]$, so that she is willing to implement at a positive rate. The expected utility of immediate implementation at t equals

$$V(t|t) = V_0 + \alpha_1 + \left(1-\bar{\phi}_d\right)\alpha_2.$$

This is exactly the same as the expected utility of waiting until $t + \Delta t$ to implement, when $d^*(t) = \frac{\lambda e_{\max}}{1-\bar{\phi}_d}$ for $t \geq X_d$,

$$V(t+\Delta t|t) = V_0 + \alpha_1 + \int_t^{t+\Delta t}(\alpha_2 - \delta(s-t))\,d^*(s)(1-\phi(s))\exp\left(-\int_t^s d^*(r)(1-\phi^*(r))\,dr\right)ds + \exp\left(-\int_t^{t+\Delta t} d^*(r)(1-\phi^*(r))\,dr\right)\left(\left(1-\bar{\phi}_d\right)\alpha_2 - \delta\Delta t\right).$$

Using that $\phi^*(r) = \bar{\phi}_d$ for $r \geq X_d$, this simplifies further to

$$V(t+\Delta t|t) = V_0 + \alpha_1 + \left(1-\exp\left(-d^*(t)\left(1-\bar{\phi}_d\right)\Delta t\right)\right)\left(\alpha_2 - \frac{\delta}{d^*(t)\left(1-\bar{\phi}_d\right)}\right) + \exp\left(-d^*(t)\left(1-\bar{\phi}_d\right)\Delta t\right)\left(\left(1-\bar{\phi}_d\right)\alpha_2\right),$$

where

$$\frac{\delta}{d^*(t)\left(1-\bar{\phi}_d\right)} = \frac{\delta}{\lambda e_{\max}} = \bar{\phi}_d\alpha_2.$$

Hence, we find indeed that

$$V(t+\Delta t|t) = V_0 + \alpha_1 + \left(1-\bar{\phi}_d\right)\alpha_2.$$

and thus the mixing strategy is an equilibrium strategy.

Strategy of unsuccessful player: Check that the unsuccessful agent's choice of effort $e^*(t) = e_{\max}$ is optimal by noting that $V^S(t) - V^U(t) > \frac{c}{\lambda}$ for all t provided $T < X_d + Z$ and $V^S(t) - V^U(t) = \frac{c}{\lambda}$ for $t \in [0, X_d]$ when $T = X_d + Z$. The argument that an unsuccessful agent will not decide to implement is again the same as in Case 1 of Proposition 1. The expected utility when following the equilibrium strategy exceeds the expected utility when exerting no effort, but delaying implementation, which exceeds the expected utility of implementing the project immediately.

iii) **Case 3:** $X_d + Z < T \leq Y_d + Z$

Notice first that the subgames for $t \geq T - Z$ are identical to those described above in case 2 for $t \geq X_d$ and the proof is identical. Turning to $t < T - Z$. For $t \in [0, T - Z]$, the continuation value for the successful individual is

$$V^S(t) = V_0 + \alpha_1 + \left(1 - \bar{\phi}_d\right)\alpha_2 - \delta(T - Z - t)$$

and for the unsuccessful agent

$$V^U(t) = \left(1 - \exp\left(-\lambda \int_t^{T-Z} e^*(s)\,ds\right)\right) V^S(T - Z) + \exp\left(-\lambda \int_t^{T-Z} e^*(s)\,ds\right) V^U(T - Z) - \frac{c}{\lambda}\left(1 - \exp\left(-\lambda \int_t^{T-Z} e^*(s)\,ds\right)\right) - \delta(T - Z - t).$$

Hence, the difference equals

$$V^S(t) - V^U(t) = \frac{c}{\lambda} \text{ for } 0 \leq t \leq T - Z.$$

Strategy of successful player: This corresponds to the proof in Case 2 of Proposition 1. Check that the successful individual's decision strategy $d^*(t) = 0$ is optimal by noting that $V^S(t) \geq V_0 + \alpha_1 + (1 - \phi^*(t))\alpha_2$ when $\phi^*(t) = \exp\left(-\lambda \int_0^t e^*(s)\,ds\right) \geq \bar{\phi}_d + \frac{\delta}{\alpha_2}(X_d - t)$, which is true given the equilibrium effort strategy specified.

Strategy of unsuccessful player: Check that the unsuccessful agent is indifferent about the level of exerted effort for all $t \in [0, T - Z]$, since $V^S(t) - V^U(t) = \frac{c}{\lambda}$. An unsuccessful agent will not implement the project provided that

$$V^U(t) > V_0 + (1 - \phi^*(t))\alpha_1$$

$\Leftrightarrow$

$$V_0 + \alpha_1 + \left(1 - \bar{\phi}_d\right)\alpha_2 - \frac{c}{\lambda} - \delta(X_d - t) > V_0 + \alpha_1 + (1 - \phi^*(t))\alpha_2 - (\phi^*(t)\alpha_1 + (1 - \phi^*(t))\alpha_2).$$

We know from the successful player's strategy that

$$\left(1 - \bar{\phi}_d\right)\alpha_2 - \delta(X_d - t) \geq (1 - \phi^*(t))\alpha_2.$$

Hence, it remains to show that

$$\phi^*(t)\alpha_1 + (1 - \phi^*(t))\alpha_2 > \frac{c}{\lambda}$$

which is true because $\bar{\phi}\alpha_1 + \left(1 - \bar{\phi}\right)\alpha_2 = \frac{c}{\lambda}$ and $\phi^*(t) > \bar{\phi}_d > \bar{\phi}$.

iv) **Case 4:** $T > Y_d + Z$

Here again we note that all subgames starting from $t = T - Y_d + Z$ are encompassed by the proof of Case 3 above, and the continuation values at $t = T - Y_d + Z$ are

$$\begin{aligned} V^S(t) &= V_0 + \alpha_1 \\ V^U(t) &= V_0 + \alpha_1 - \frac{c}{\lambda} \end{aligned}$$

which are exactly the same continuation values as in **Case 3** of Proposition 1 for $t = T - Y$. In this case the strategies and the proof for the subgames $t < T - (Y_d + Z)$ are identical to that for **Case 3** of Proposition 1 for

$t < T - Y$. □

For $T < X_d$, the equilibrium strategies are exactly like before. For $T \geq X_d$, the marginal value of a breakthrough at and close to the deadline is strictly greater than $\frac{c}{\lambda}$, unlike in the small incentives case. This also continues to be the case for all longer deadlines. As the length of the game T increases, however, the incentives for effort at a given time decrease. To see this, consider the incentives for effort at $t = 0$ which are given by

$$V^S(0) - V^U(0) = \frac{c}{\lambda} + \bar{\phi}_d \left(V^S(X_d) - V^U(X_d) - \frac{c}{\lambda} \right).$$

The only part of the expression which changes with T, is $V^U(X_d)$ since all the other terms above are constants and

$$V^S(X_d) = V^S(T) = V_0 + \alpha_1 + \left(1 - \bar{\phi}_d\right)\alpha_2.$$

$V^U(X_d)$ can be rewritten as

$$V^U(X_d) = \exp\left(-2\lambda e_{\max}(T - X_d)\right) V^U(T) + \left(1 - \exp\left(-2\lambda e_{\max}(T - X_d)\right)\right)[V_0 + \alpha_1 + \frac{\left(1 - \bar{\phi}_d\right)\alpha_2}{2} - \left(ce_{\max} + \delta\right)\frac{1}{2\lambda e_{\max}}].$$

This is a weighted sum of the expected payoff conditional on either producing a breakthrough or the other agent implementing the project prior to the deadline and the payoff from being unsuccessful at the deadline. Both of these payoffs are independent of T and it is only the relative likelihood of each which is affected by T. The likelihood of being unsuccessful at the deadline $\exp\left(-2\lambda e_{\max}(T - X_d)\right)$ decreases in T. Hence, the continuation value of being unsuccessful at X_d is increasing in T. There exists a deadline $T = X_d + Z$ where $V^S(X_d) - V^U(X_d) = \frac{c}{\lambda}$ and an agent is indifferent about exerting effort at $t = 0$, so $V^S(0) - V^U(0) = \frac{c}{\lambda}$. For T larger than $X_d + Z$, that is case iii) above, maximal effort by unsuccessful agents can no longer be sustained throughout the entire game. In equilibrium, an unsuccessful agent reduces her average effort intensity before $t_d = T - Z$ such that $\phi(t_d) = \bar{\phi}_d$, while the successful agent fully delays. For T larger than $Y_d + Z$, successful agents will no longer prefer to delay their implementation decision at the beginning of the game, exactly as in the case with small incentives.

B.3 Uniqueness

In this section, we prove uniqueness of our proposed equilibrium. We first provide a more general description of our model and then state a sequence of lemmas that we use to prove Proposition 3 which establishes the uniqueness of our equilibria described in Proposition 2 and Proposition 8.

B.3.1 General Model Description

To establish uniqueness we introduce some additional notation that describes the agents' strategies. Define the decision strategy of the unsuccessful agent by $\mu(t)$ which is the conditional probability of implementing a project with no breakthroughs produced by time t given that the other agent does not implement prior to t. Define the decision strategy of the successful agent by $\rho(t)$ as the conditional probability of implementing the project with one breakthrough produced by time t given that the other agent does not implement prior to t. Define $e(t)$ as the effort strategy of the unsuccessful agent. Define $\sigma(t)$ as the conditional probability of being unsuccessful by time t given that the other agent does not implement prior to t. Hence, $1 - \sigma(t)$ is the conditional probability of being successful by that time. $\sigma(t)$ changes over time according to

$$\frac{d\sigma}{dt} = -\lambda e(t)\left(\sigma(t) - \mu(t)\right)$$

The effort strategy of the unsuccessful agent influences $\frac{d\sigma}{dt}$. The implementation decision strategy of an unsuccessful agent influences both $\frac{d\sigma}{dt}$ and $\mu(t)$. The implementation decision strategy of the successful agent controls

$\rho(t)$. The following relationships between these functions hold

$$\mu(t) + \rho(t) \leq 1$$

and

$$\mu(t) \leq \sigma(t)$$

and

$$\rho(t) \leq 1 - \sigma(t)$$

We will use the notation $\tilde{}$ (tilde) to denote the strategies of the other player and * (star) to denote equilibrium strategies. The Bayesian belief $\phi(t)$ at a time t that the other agent is unsuccessful conditional on the project not being implemented prior to that time is

$$\phi(t) = \frac{\tilde{\sigma}(t) - \tilde{\mu}(t)}{1 - \tilde{\mu}(t) - \tilde{\rho}(t)}$$

A strategy for an agent maps into a path for $e(t), \sigma(t), \mu(t), \rho(t)$. We restrict our attention to strategies which result in piecewise continuously differentiable functions of $e(t)$, $\sigma(t)$, $\mu(t)$ and $\rho(t)$. Clearly, given the nature of the model $\frac{d\sigma}{dt} \leq 0$ since agents do not lose or destroy breakthroughs and $\frac{d\mu}{dt}, \frac{d\rho}{dt} \geq 0$ since deciding to implement the project is irreversible. The upper bound on $e(t)$ also ensures that $\sigma(t)$ is continuous.

We have assumed that $\rho(t)$ and $\mu(t)$ are continuous and differentiable at all but a finite number of points. Denote the set of points where the strategy is discontinuous by $\chi_\rho = \{t_1^\rho, ...t_\nu^\rho\}$, $\chi_\mu = \{t_1^\mu, ...t_\nu^\mu\}, \tilde{\chi}_\rho = \left\{\tilde{t}_1^\rho, ...\tilde{t}_{\tilde{\nu}}^\rho\right\}$, $\tilde{\chi}_\mu = \left\{\tilde{t}_1^\mu, ...\tilde{t}_{\tilde{\nu}}^\mu\right\}$ and define $\chi = \chi_\rho \cup \chi_\mu$ and $\tilde{\chi} = \tilde{\chi}_\rho \cup \tilde{\chi}_\mu$. Also, define

$$D_\rho(t) = \lim_{s \to t^+} \rho(s) - \lim_{r \to t^-} \rho(r)$$

and

$$D_\mu(t) = \lim_{s \to t^+} \mu(s) - \lim_{r \to t^-} \mu(r)$$

These are non-zero only at points in χ_ρ and χ_μ respectively and represent the probability that a decision to implement at that moment conditional on the other agent not implementing the project prior to that time. The objective function of the agent is:

$$\max_{e(t),\mu(t),\rho(t)} V_0 + \int_0^T \left[\alpha_1 + \left(\frac{1-\tilde{\sigma}(t)-\tilde{\rho}(t)}{1-\tilde{\mu}(t)-\tilde{\rho}(t)}\right)\alpha_2\right]\frac{d\rho}{dt}\left[1-\tilde{\mu}(t)-\tilde{\rho}(t)\right]dt$$
$$+\int_0^T \left(\frac{1-\tilde{\sigma}(t)-\tilde{\rho}(t)}{1-\tilde{\mu}(t)-\tilde{\rho}(t)}\right)\alpha_1\frac{d\mu}{dt}\left[1-\tilde{\mu}(t)-\tilde{\rho}(t)\right]dt$$
$$+\int_0^T \left[\alpha_1 + \left(\frac{1-\sigma(t)-\rho(t)}{1-\mu(t)-\rho(t)}\right)\alpha_2\right]\frac{d\tilde{\rho}}{dt}\left[1-\mu(t)-\rho(t)\right]dt$$
$$+\int_0^T \left(\frac{1-\sigma(t)-\rho(t)}{1-\mu(t)-\rho(t)}\right)\alpha_1\frac{d\tilde{\mu}}{dt}\left[1-\mu(t)-\rho(t)\right]dt$$
$$+\sum_{t\in\chi\cap\tilde{\chi}}\left[1-\tilde{\mu}(t)-\tilde{\rho}(t)\right]\left\{\begin{array}{c} D_\rho(t)\left[\alpha_1+\left(\frac{1-\tilde{\sigma}(t)-\tilde{\rho}(t)}{1-\tilde{\mu}(t)-\tilde{\rho}(t)}\right)\alpha_2\right] \\ +D_\mu(t)\left(\frac{1-\tilde{\sigma}(t)-\tilde{\rho}(t)}{1-\tilde{\mu}(t)-\tilde{\rho}(t)}\right)\alpha_1\end{array}\right\}$$
$$+\sum_{t\in\chi\cap\tilde{\chi}}\left[1-\mu(t)-\rho(t)-D_\rho(t)-D_\mu(t)\right]\left\{\begin{array}{c} D_{\tilde{\rho}}(t)\left[\alpha_1+\left(\frac{1-\sigma(t)-\rho(t)-D_\rho(t)}{1-\mu(t)-\rho(t)-D_\rho(t)-D_\mu(t)}\right)\alpha_2\right] \\ D_{\tilde{\mu}}(t)\left(\frac{1-\sigma(t)-\rho(t)-D_\rho(t)}{1-\mu(t)-\rho(t)-D_\rho(t)-D_\mu(t)}\right)\alpha_1\end{array}\right\}$$
$$+\sum_{t\in\chi\backslash\tilde{\chi}}\left[1-\tilde{\mu}(t)-\tilde{\rho}(t)\right]\left\{D_\rho(t)\left[\alpha_1+\left(\frac{1-\tilde{\sigma}(t)-\tilde{\rho}(t)}{1-\tilde{\mu}(t)-\tilde{\rho}(t)}\right)\alpha_2\right]+D_\mu(t)\left(\frac{1-\tilde{\sigma}(t)-\tilde{\rho}(t)}{1-\tilde{\mu}(t)-\tilde{\rho}(t)}\right)\alpha_1\right\}$$
$$+\sum_{t\in\tilde{\chi}\backslash\chi}\left[1-\mu(t)-\rho(t)\right]\left\{D_{\tilde{\rho}}(t)\left[\alpha_1+\left(\frac{1-\sigma(t)-\rho(t)}{1-\mu(t)-\rho(t)}\right)\alpha_2\right]+D_{\tilde{\mu}}(t)\left(\frac{1-\sigma(t)-\rho(t)}{1-\mu(t)-\rho(t)}\right)\alpha_1\right\}$$
$$+\left(1-\mu(T)-\rho(T)\right)\left(1-\tilde{\mu}(T)-\tilde{\rho}(T)\right)\left[\alpha_1+\left(\frac{1-\tilde{\sigma}(T)-\tilde{\rho}(T)}{1-\tilde{\mu}(T)-\tilde{\rho}(T)}\right)\alpha_2\right]$$
$$-\delta T\left(1-\mu(T)-\rho(T)\right)\left(1-\tilde{\mu}(T)-\tilde{\rho}(T)\right)$$
$$-\int_0^T \delta\left(1-\mu(t)-\rho(t)\right)\left(1-\tilde{\mu}(t)-\tilde{\rho}(t)\right)dt$$
$$-c\left(1-\tilde{\mu}(T)-\tilde{\rho}(T)\right)\left(\sigma(T)-\mu(T)\right)\int_0^T e(t)\,dt$$
$$-c\int_0^T e(t)\left(1-\tilde{\mu}(t)-\tilde{\rho}(t)\right)\left(\sigma(t)-\mu(t)\right)dt$$

$$\begin{aligned} s.t.\ \rho(t) &\leq 1-\sigma(t),\ \mu(t)\leq\sigma(t) \\ \sigma(0) &= 1,\ \frac{d\sigma}{dt}=-\lambda e(t)\left(\sigma(t)-\mu(t)\right)\end{aligned}$$

where $\tilde{\mu}(t), \tilde{\rho}(t), \tilde{\sigma}(t)$ denote the strategy of the other player. To simplify notation we will continue using the definition of $\phi(t) = \frac{\tilde{\sigma}(t)-\tilde{\mu}(t)}{1-\tilde{\mu}(t)-\tilde{\rho}(t)}$.

We are interested in perfect Bayesian equilibria of the model so strategies must be an equilibrium for all subgames starting at each time t. We describe these by writing out the problem in terms of continuation values for the successful $V^S(t)$ and unsuccessful agent $V^U(t)$ upon reaching time t. Also, write $\tilde{\rho}(s|t)$ $\tilde{\mu}(s|t)$ for the perceived probabilities that a successful and unsuccessful agent implement the project at $s \geq t$ given the agent is at t. At on-equilibrium times these are $\tilde{\rho}(s|t) = \frac{\tilde{\rho}(s)-\tilde{\rho}(t)}{1-\tilde{\rho}(t)}$ and $\tilde{\mu}(s|t) = \frac{\tilde{\mu}(s)-\tilde{\mu}(t)}{1-\tilde{\mu}(t)}$. However, if an off-equilibrium time is reached then this is no longer the case and separate equilibrium strategies and beliefs must be specified for these subgames. As we will show below the set of symmetric perfect Bayesian equilibria we focus on involve all times being reached with positive probability.

The continuation value of being successful is just the payoff from implementing the optimal stopping policy

$\widehat{t}_\rho$ from that moment forward:

$$\begin{aligned} V^S(t) &= \max_{\widehat{t}\in[t,T]} V_0 + \alpha_1 + \int_t^{\widehat{t}} (\alpha_2 - \delta(r-t)) \frac{d\tilde{\rho}(r|t)}{dt} dr + \int_t^{\widehat{t}} (-\delta(r-t)) \frac{d\tilde{\mu}(r|t)}{dt} dr \\ &+ \sum_{\substack{r\in\tilde{\chi} \\ t<r<\widehat{t}}} D_{\tilde{\rho}}(r|t)(\alpha_2 - \delta(r-t)) + D_{\tilde{\mu}}(r|t)(-\delta(r-t)) \\ &+ \left(1 - \tilde{\mu}(\widehat{t}|t) - \tilde{\rho}(\widehat{t}|t)\right)\left((1-\phi^*(\widehat{t}|t))\alpha_2 - \delta\left[\widehat{t}-t\right]\right) \end{aligned} \tag{5}$$

where $\hat{t}^*_\rho(t)$ is the optimizer or set of optimizers of equation 5. The implementation decision strategy is optimal provided that:

$$\begin{aligned} \lim_{s\to r^+} \rho^*(s|t) &= \sigma(r) \text{ if } r = \hat{t}^*_\rho(t) \\ \frac{d\rho^*(r|t)}{dt} &\geq 0 \text{ or } D_{\rho^*}(r|t) \geq 0 \text{ if } r \in \hat{t}^*_\rho(t) \\ \frac{d\rho^*(r|t)}{dt} &= 0 \text{ if } r \notin \hat{t}^*_\rho(t) \end{aligned}$$

and these conditions ensure ρ satisfies the adding up constraint:

$$\int_{\hat{t}^*_\rho(t)} \frac{d\rho^*(t)}{dt} dt = \sigma\left(\max\left(\hat{t}^*(t)\right)\right) - \rho(t) + (\sigma(T) - \rho(T)) \times \mathbf{1}\left(\max\left(\hat{t}^*(t)\right) = T\right)$$

The payoff from being unsuccessful is defined by a joint effort and stopping problem given by:

$$\begin{aligned} V^U(t) = &\max_{\substack{\widehat{t}\in[t,T] \\ e(r|t) \text{ for } r\in[t,\widehat{t}]}} V_0 + \int_t^{\widehat{t}} \left[\alpha_1 - \delta(r-t) - c\int_t^r e(w|t)\,dw\right] \frac{d\tilde{\rho}(r|t)}{dt} \left[1 - \exp\left(-\lambda \int_t^r e(w|t)\,dw\right)\right] dr \\ &+ \int_t^{\widehat{t}} \left[-\delta(r-t) - c\int_t^r e(w|t)\,dw\right] \frac{d\tilde{\mu}(r)}{dt} \left[1 - \exp\left(-\lambda \int_t^r e(w|t)\,dw\right)\right] dr \\ &+ \int_t^{\widehat{t}} \left[V^S(t) - \delta(r-t) - c\int_t^r e(w|t)\,dw\right] \lambda e(r) \exp\left(-\lambda \int_t^r e(w|t)\,dw\right) (1 - \tilde{\mu}(r|t) - \tilde{\rho}(r|t))\,dr \\ &+ \sum_{\substack{r\in\tilde{\chi}\setminus\chi \\ t<r<\widehat{t}_\mu(t)}} \frac{1-\mu(r)-\sigma(r)}{1-\mu(t)-\sigma(t)} \begin{bmatrix} D_{\tilde{\rho}}(r|t)\left(\alpha_1 - \delta(r-t) - c\int_t^r e(w|t)\,dw\right) \\ +D_{\tilde{\mu}}(r|t)\left(-\delta(r-t) - c\int_t^r e(w|t)\,dw\right) \end{bmatrix} \\ &+ \left(1 - \exp\left(-\int_t^{\widehat{t}} \lambda e(r)\,dr\right)\right) \left(1 - \tilde{\mu}(\widehat{t}|t) - \tilde{\rho}(\widehat{t}|t)\right) \left[\left(1 - \phi^*(\widehat{t}|t)\right)\alpha_1 - \delta\left(\widehat{t} - t\right) - c\int_t^{\widehat{t}} e(r|t)\,dr\right] \end{aligned} \tag{6}$$

where $\hat{t}^*_\mu(t)$ is the optimizer or set of optimizers of equation 6. The condition for the effort strategy profile to be an equilibrium satisfies

$$e^*(t) = \arg\max_{e\in[0,1]} \lambda e\left(V^S(t) - V^U(t)\right) - ce$$

and the implementation decision strategy is an equilibrium provided that

$$\begin{aligned} \lim_{s\to r^+} \mu^*(s|t) &= 1 - \sigma(r) \text{ if } r = \hat{t}^*_\mu(t) \\ \frac{d\mu^*(r|t)}{dt} &\geq 0 \text{ or } D_{\rho^*}(r|t) \geq 0 \text{ if } r \in \hat{t}^*_\mu(t) \\ \frac{d\mu^*(r|t)}{dt} &= 0 \text{ if } r \notin \hat{t}^*_\mu(t) \end{aligned}$$

where $\hat{t}^*(t)$ solves (6) the unsuccessful agent's effort and stopping problem. These conditions also ensure that it satisfies the adding up constraint

$$\int\limits_{\hat{t}^*_\mu(t)} \frac{d\mu^*(t)}{dt}dt = 1-\sigma\left(\max\left(\hat{t}^*_\mu(t)\right)\right) - \mu(t) + (1-\sigma(T)-\mu(T)) \times \mathbf{1}\left(\max\left(\hat{t}^*_\mu(t)\right) = T\right)$$

A symmetric perfect Bayesian equilibrium may be described by a tuple $(e^*(t), \rho^*(t), \mu^*(t), \phi^*(t))$ if $\rho^*(t) + \mu^*(t) < 1$ for all $t < T$, where $\phi^*(t)$ is the Bayesian belief an agent has at time t that the other agent is unsuccessful conditional on the project not having been implemented prior to that time. If $\exists t' < T : \rho^*(t') + \mu^*(t') = 1$ then it must also include off-equilibrium strategies and beliefs $(e^*(r|t), \rho^*(r|t), \mu^*(r|t), \phi^*(r|t))$ for all times t where $\rho^*(t) + \mu^*(t) = 1$ which themselves are equilibria of those subgames, where $\phi^*(r|t)$ is the Bayesian belief an agent has at time r that the other agent is unsuccessful conditional on no project implementation prior to that time in a subgame starting at time t. We now rule out some types of decision strategies at on-equilibrium times by the unsuccessful agent. The following lemma rules out a continuously increasing $\mu^*(t)$.

Lemma 1. *$\nexists \mu^*(t), r > 0, \varepsilon > 0 : \frac{d\mu^*(t)}{dt} > 0$ for $t \in [r-\varepsilon, r]$.*

Proof. Suppose not and $\exists \mu^*(t) : \frac{d\mu^*(t)}{dt} > 0$ for $t \in [r-\varepsilon, r]$. If an unsuccessful agent weakly prefers to implement then a successful agent strictly prefers to implement hence

$$\rho^*(t) = \sigma^*(t) \text{ for } t \in (r-\varepsilon, r)$$

and

$$\phi^*(t) = 1 \text{ for } t \in (r-\varepsilon, r)$$

if not then $\exists r' > t$ such that

$$\begin{aligned} V_0 + \alpha_1 + (1-\phi^*(t))\alpha_2 \leq\ & V_0 + \alpha_1 + (1-\tilde{\mu}^*(r'|t) - \tilde{\rho}^*(r'|t))\left[(1-\phi^*(r'))\alpha_2 - \delta(r'-t)\right] \\ & - \delta\left[\int_t^{r'} (y-t)\left[\frac{d\tilde{\mu}^*(y|t)}{dy} + \frac{d\tilde{\rho}^*(y|t)}{dy}\right]dy + \sum_{\substack{y\in\tilde{X} \\ t<y<r'}} (y-t)\left[D_{\tilde{\mu}}(y|t) + D_{\tilde{\rho}}(y|t)\right]\right] \\ & + \alpha_2\left[\int_t^{r'} \frac{d\tilde{\rho}^*(r'|t)}{dy}dy + \sum_{\substack{y\in\tilde{X} \\ t<y<r'}} D_{\tilde{\rho}}(y|t)\right] \end{aligned}$$

This can be rewritten as

$$\begin{aligned} & \delta\left[\int_t^{r'} (y-t)\left[\frac{d\tilde{\mu}^*(y|t)}{dy} + \frac{d\tilde{\rho}^*(y|t)}{dy}\right]dy + \sum_{\substack{y\in\tilde{X} \\ t<y<r'}} \left[D_{\tilde{\mu}}(y|t) + D_{\tilde{\rho}}(y|t)\right](-\delta(y-t)) + (r'-t)(1-\tilde{\mu}^*(r'|t))\right] \\ & \leq \alpha_2\left[(1-\tilde{\mu}^*(r'|t) - \tilde{\rho}^*(r'|t))(1-\phi^*(r')) - (1-\phi^*(t)) + \int_t^{r'} \frac{d\tilde{\rho}^*(r'|t)}{dy}dy + \sum_{\substack{y\in\tilde{X} \\ t<y<r'}} D_{\tilde{\rho}}(y|t)\right]. \end{aligned}$$

However, if this inequality holds then an unsuccessful agent could do strictly better by delaying implementation until r' since the comparison of payoffs would result in the same expression except with α_1 replacing α_2. The inequality would then be strict and this would be a contradiction of the unsuccessful agent mixing at t. Hence, $\phi^*(t) = 1$ for $t \in (r-\varepsilon, r)$. However, if $\phi^*(t) = 1$ then an uniformed agent can do strictly better by delaying implementation and exerting effort $e_{\max}$ over a period of time since

$$\lambda e_{\max}\alpha_1 > c e_{\max} + \delta$$

by assumption. □

The following lemma rules out a jump in the implementation decision function $\mu^*(t)$ at a time on the equilibrium path as long as that jump does not occur when both types, successful and unsuccessful, decide to implement with certainty at that instant.

Lemma 2. *$\nexists \mu^*(t), 0 < s < T : D_{\mu^*}(s) > 0$ and $\mu^*(s) + \rho^*(s) < 1$*

Proof. Suppose not and $\exists s : D_{\mu^*}(s) > 0$ and $\mu^*(s) + \rho^*(s) < 1$. As above, this implies

$$\lim_{r \to s^+} \phi^*(r) = 1$$

by the same reasoning as before. Hence, an unsuccessful agent at s can do better than deciding to implement immediately by delaying and putting in effort since

$$\lambda e_{\max} \alpha_1 > c e_{\max} + \delta$$

by assumption. □

The only equilibria involving $D_{\mu^*}(s) > 0$ also have $\mu^*(s) + \rho^*(s) = 1$ whereby beliefs at times later than s are off the equilbrium path. In this case it may be possible to support unsuccessful agents implementing the project with appropriately specified off-equilibrium-path beliefs. However, we will exclude this type of equilibrium as we feel for all intents and purposes that it is equivalent to imposing a deadline at time s. We thus continue the analysis under the assumption that $\mu^*(t) = 0$ for all t. This also means that all times are reached on the equilibrium path. The following lemma rules out jumps in the implementation decision function of the successful type $\rho^*(t)$.

Lemma 3. *$\nexists \rho^*(t), 0 < t^\wedge < T : D_{\rho^*}(t^\wedge) > 0$.*

Proof. We proceed with a proof by contradiction. Say there is an equilibrium with a mass point at a time $t^\wedge$ where a mass of $D_{\rho^*}(t^\wedge) = (1 - \phi^*(t^\wedge))\beta > 0$ decisions to implement are made. For this to be the case then $\phi < 1$. If $\phi^* = 1$ $\rho^*(t) = \sigma^*(t)$ successful agents may only decide to implement at the rate at which unsuccessful agents are becoming successful. Consider $\lim_{t \to t^{\wedge -}} e^*(t)$ and $\lim_{t \to t^{\wedge +}} e^*(t)$. For there to be a mass point the following conditions need to hold for an agent not to implement the project earlier or later:

$$\lim_{t \to t^{\wedge -}} \lambda e^*(t) \alpha_2 (1 - \phi^*(t)) > \delta - \varepsilon \text{ for any } \varepsilon > 0$$

and

$$\lim_{t \to t^{\wedge +}} \lambda e^*(t) \alpha_2 (1 - \phi^*(t)) < \delta + \varepsilon \text{ for any } \varepsilon > 0$$

The first of these inequalities implies that a successful agent will be willing not to implement in the neighborhood immediately prior to $t^\wedge$. The second guarantees that a successful agent cannot do better by waiting at time $t^\wedge$. Also note that $\lim_{t \to t^{\wedge +}} \phi^*(t^\wedge) > \lim_{t \to t^{\wedge -}} \phi^*(t^\wedge)$ due to the mass point. This implies there is a discontinuous change in the effort level at $t^\wedge$ if the above two conditions are to be satisfied. We show that this cannot be maintained in equilibrium. We can write the continuation value from being unsuccessful at time $t = t^\wedge - \Delta t$ as:

$$\begin{aligned}
V^U(t^\wedge - \Delta t) = \int_{t^\wedge - \Delta t}^{t^{\wedge -}} &\left(\frac{\frac{d\sigma^*(s|t^\wedge - \Delta t)}{ds}}{1 - \sigma^*(s|t^\wedge - \Delta t)} V^S(t) + (V_0 + \alpha_1) \frac{\frac{d\tilde{\rho}^*(s|t^\wedge - \Delta t)}{ds}}{1 - \tilde{\rho}^*(s|t^\wedge - \Delta t)} - c e^*(s) - \delta \right) \times \\
&(1 - \sigma^*(s|t^\wedge - \Delta t))(1 - \tilde{\rho}^*(s|t^\wedge - \Delta t))\, ds \\
&+ (1 - \sigma^*(t^\wedge | t^\wedge - \Delta t)) D_{\tilde{\rho}^*}(t^\wedge | t^\wedge - \Delta t) \left(V_0 + \alpha_1 - \delta \Delta t - c \int_{t^\wedge - \Delta t}^{t^\wedge} e^*(s)\, ds \right) \\
&\left(\begin{array}{c} \int_{t^{\wedge +}}^{t^\wedge + \Delta t} \left(\frac{\frac{d\sigma^*(s|t^\wedge - \Delta t)}{ds}}{1 - \sigma^*(s|t^\wedge - \Delta t)} V^S(t) + (V_0 + \alpha_1) \frac{\frac{d\tilde{\rho}^*(s|t^\wedge - \Delta t)}{ds}}{1 - \tilde{\rho}^*(s|t^\wedge - \Delta t)} - c e^*(s) - \delta \right) \times \\ (1 - \sigma^*(s|t^\wedge - \Delta t))(1 - \tilde{\rho}^*(s|t^\wedge - \Delta t))\, ds \\ + (1 - \sigma^*(t^\wedge + \Delta t | t^\wedge - \Delta t))(1 - \tilde{\rho}^*(t^\wedge + \Delta t | t^\wedge - \Delta t)) V^U(t^\wedge + \Delta t) \end{array} \right)
\end{aligned}$$

Where Δt may always be chosen small enough such that there are no other points of discontinuity of $\rho^*(t)$ for $t \in [t\hat{} - \Delta t, t\hat{} + \Delta t]$ other than at $t = t\hat{}$. Now consider moving a unit of effort from $t\hat{} - \varepsilon$ to $t\hat{} + \varepsilon$ by augmenting the strategy $e^*(t)$ as follows:

$$\begin{aligned} e^{**}(t) &= e^*(t) - \varepsilon \text{ for } t \in [t\hat{} - \Delta t, t\hat{}) \\ e^{**}(t) &= e^*(t) + \varepsilon \text{ for } t \in [t\hat{}, t\hat{} + \Delta t] \end{aligned}$$

The strategies are piecewise continuous so we can always find a Δt such that they are continuous over the intervals $[t - \Delta t, t)$ and $(t, t + \Delta t]$. Using a Taylor series expansion

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{V^U(t\hat{} - \Delta t|e^{**}) - V^U(t\hat{} - \Delta t|e^*)}{\Delta t} = -\varepsilon\left(\lambda\left(\lim_{t\to t\hat{}-} V^S(t)\right) - c\right) \\ + \lambda\varepsilon D_{\rho^*}(t\hat{}|t\hat{} - \Delta t)(V_0 + \alpha_1) \\ + \varepsilon(1 - D_{\rho^*}(t\hat{}|t\hat{} - \Delta t))\left(\lambda\left(\lim_{t\to t\hat{}+} V^S(t)\right) - c\right) - O(\Delta t) \end{aligned}$$

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{V^U(t\hat{} - \Delta t|e^{**}) - V^U(t\hat{} - \Delta t|e^*)}{\Delta t} \geq -\varepsilon(\lambda(V_0 + \alpha_1 + (1 - \phi(t\hat{}))\alpha_2) - c) \\ + \lambda\varepsilon D_{\rho^*}(t\hat{}|t\hat{} - \Delta t)(V_0 + \alpha_1) \\ + \varepsilon(1 - D_{\rho^*}(t\hat{}|t\hat{} - \Delta t))(\lambda(V_0 + \alpha_1 + (1 - \phi(t\hat{}) - D_{\rho^*}(t\hat{}|t\hat{} - \Delta t))\alpha_2) - c) - O(\Delta t) \end{aligned}$$

Define the right-hand side by R then

$$\begin{aligned} R &= -D_{\rho^*}(t\hat{}|t\hat{} - \Delta t)\lambda\alpha_2\varepsilon + D_{\rho^*}(t\hat{}|t\hat{} - \Delta t)c\varepsilon - O(\Delta t) \\ &= \varepsilon D_{\rho^*}(t\hat{}|t\hat{} - \Delta t)(c - \lambda\alpha_2) - O(\Delta t) \\ &> 0 \end{aligned}$$

Hence, there exists $\Delta t > 0$ such that this change in strategy is profitable which is a contradiction that the original effort $e^*(t)$ is optimal and can be part of an equilibrium. □

This along with the earlier lemmas that unsuccessful individuals do not decide to implement implies that $\phi^*(t)$, $\rho^*(t)$ and $\sigma^*(t)$ are all continuous.

Lemma 4. *$V^S(t)$ is continuous.*

Proof. The continuity of $\phi^*(t)$, $\rho^*(t)$ and $\sigma^*(t)$ ensures that

$$\begin{aligned} f\left(t, \widehat{t}\right) = V_0 + \alpha_1 + \int_t^{\widehat{t}} (\alpha_2 - \delta(r - t)) \frac{d\tilde{\rho}^*(r|t)}{dt} dr \\ + \left(1 - \tilde{\rho}^*\left(\widehat{t}|t\right)\right)\left(\left(1 - \phi^*\left(\widehat{t}|t\right)\right)\alpha_2 - \delta\left[\widehat{t} - t\right]\right) \end{aligned}$$

is continuous in $\hat{t}$. Hence, $V^S(t) = \max_{\hat{t}\in[t,T]} f(t)$ is continuous in t by the theorem of the maximum (Berge 1963). □

Lemma 5. *$V^U(t)$ is continuous.*

Proof. The continuity of $\phi^*(t)$, $\rho^*(t)$ and $\sigma^*(t)$ ensures that

$$f(t, e(r|t)) = V_0 + \int_t^T \left(\alpha_1 - \delta(r-t) - c\int_t^r e(w|t)\,dw\right)\frac{d\tilde{\rho}^*(r|t)}{dt}\left[1 - \exp\left(-\lambda\int_t^r e(w|t)\,dw\right)\right]dr$$
$$+ \int_t^T \left(V^S(t) - \delta(r-t) - c\int_t^r e(w|t)\,dw\right)\lambda e(r)\exp\left(-\lambda\int_t^r e(w|t)\,dw\right)[1 - \tilde{\rho}^*(r|t)]\,dr$$
$$+ \left(1 - \exp\left(-\int_t^T e(r)\,dr\right)\right)(1 - \tilde{\rho}^*(T|t))\left[(1 - \phi^*(T|t))\alpha_1 - \delta(T-t) - c\int_t^T e(r|t)\,dr\right]$$

is continuous in $e(r|t)$. Hence,

$$V^S(t) = \max_{e(r|t)\in C_1([t,T],[0,e_{\max}])} f(t, e(r|t))$$

where $C_1([t,T],[0,e_{\max}])$ are piecewise continuous functions with domain $[t,T]$ and range $[0,e_{\max}]$ that are continuous in t by the theorem of the maximum (Berge 1963). □

Lemma 6. *Suppose $\rho^*(t)$ and $e^*(t)$ constitute equilibrium strategies and $\exists s, \Delta s > 0 : \frac{d\rho^*(t)}{dt} > 0$ and $0 < e^*(t) \le e_{\max}$ for $t \in [s, s+\Delta s]$, then $e^*(t) = \frac{\delta}{\lambda\phi(t)\alpha_2}$ for $t \in [s+\Delta s]$.*

Proof. We use the relation $d^*(t)(1-\phi^*(t)) = \frac{\frac{d\rho^*(t)}{dt}}{1-\rho^*(t)}$. Also, note that for $\mu^*(t) = 0$

$$\begin{aligned}
\phi(t) &= \frac{\tilde{\sigma}(t)}{1-\tilde{\rho}(t)} \\
\frac{d\phi}{dt} &= \frac{\frac{d\tilde{\sigma}}{dt} + \frac{\frac{d\tilde{\rho}}{dt}}{1-\tilde{\rho}(t)}\tilde{\sigma}(t)}{1-\tilde{\rho}(t)} \\
&= \frac{-\lambda\tilde{e}(t)\tilde{\sigma}(t) + \tilde{d}(t)(1-\phi(t))\tilde{\sigma}(t)}{1-\tilde{\rho}(t)} \\
&= \left[\tilde{d}(t)(1-\phi(t)) - \lambda\tilde{e}(t)\right]\frac{\tilde{\sigma}(t)}{1-\tilde{\rho}(t)} \\
&= \left[\tilde{d}(t)(1-\phi(t)) - \lambda\tilde{e}(t)\right]\phi(t)
\end{aligned}$$

The incentives for delaying rather than implementing are equal if the agent is mixing, that is

$$V_0 + \alpha_1 + (1-\phi^*(t))\alpha_2 = \int_t^{t+\Delta t} -\delta(r-t)\tilde{d}^*(r)(1-\phi^*(r))\exp\left(-\int_t^r \tilde{d}^*(s)(1-\phi^*(s))\,ds\right)dr$$
$$+ (V_0 + \alpha_1 + \alpha_2)\left(1 - \exp\left(-\int_t^{t+\Delta t} \tilde{d}^*(s)(1-\phi^*(s))\,ds\right)\right)$$
$$+ \exp\left(-\int_t^r \tilde{d}^*(s)(1-\phi^*(s))\,ds\right)(V_0 + \alpha_1 + (1-\phi^*(t+\Delta t))\alpha_2 - \Delta t\delta)$$

This can be rearranged to obtain

$$\delta\int_t^{t+\Delta t}(r-t)\tilde{d}^*(r)(1-\phi^*(r))\exp\left(-\int_t^r \tilde{d}^*(s)(1-\phi^*(s))\,ds\right)dr$$
$$+ \delta\Delta t\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)$$
$$= \alpha_2\left(\phi(t) - \phi(t+\Delta t)\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)\right)$$

Apply a Taylor series expansion to $\phi^*(t+\Delta t)$ and $\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)$

$$\phi^*(t+\Delta t)=\phi^*(t)-\int_t^{t+\Delta t}\left[\lambda\tilde{e}^*(s)-\tilde{d}^*(s)(1-\phi^*(s))\right]\phi^*(s)\,ds$$

$\Leftrightarrow$

$$\phi^*(t+\Delta t)=\phi^*(t)\left\{1-\Delta t\left[\lambda\tilde{e}^*(t)-\tilde{d}^*(t)(1-\phi^*(t))\right]\right\}+O\left((\Delta t)^2\right)$$

and apply it also to $\exp\left(-\int_t^{t+\Delta t}d(s)(1-\phi(s))\,ds\right)$:

$$\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)=1-\Delta t\tilde{d}^*(t)(1-\phi^*(t))+O\left((\Delta t)^2\right)$$

We combine these expressions and denote the expression inside of the brackets on the right-hand side by $R=\phi^*(t)-\phi^*(t+\Delta t)\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)$

$$R=\phi^*(t)\,\alpha_2\left\{\Delta t\left[\lambda\tilde{e}^*(t)-\tilde{d}^*(t)(1-\phi^*(t))\right]\right\}+\Delta t\tilde{d}^*(t)(1-\phi^*(t))+O\left((\Delta t)^2\right)$$

Simplifying this expression yields

$$R=\Delta t\left[\lambda\tilde{e}^*(t)\,\phi^*(t)\,\alpha_2\right]+O\left((\Delta t)^2\right)$$

Denote the left-hand side by L and apply a Taylor series expansion:

$$L=\delta\int_t^{t+\Delta t}(r-t)\,\tilde{d}^*(r)(1-\phi^*(r))\exp\left(-\int_t^r\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)dr + \delta\Delta t\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)$$

$$L=\delta\Delta t+O\left(\Delta t^2\right)$$

Equating the Δt terms from the left- and right-hand sides leads one to conclude that $\delta=\lambda\tilde{e}^*(t)\,\phi^*(t)\,\alpha_2$ □

The indifference condition implies $\lambda\tilde{e}^*(t)\,\phi^*(t)\,\alpha_2=\delta$ when $\tilde{d}^*(t)>0$ for all t.

Lemma 7. *Suppose $\rho^*(t)$, $e^*(t)$ and $\phi^*(t)$ constitute equilibrium strategies and beliefs, and $\exists s,\Delta s>0$: $\frac{d\rho^*(t)}{dt}>0,\phi^*(t)<1$ and $0<e^*(t)\le e_{\max}$ for $t\in[s,s+\Delta s]$, then $e^*(t)=e_{\max},\phi^*(t)=\bar{\phi}_d,d^*(t)=\frac{\lambda^2\alpha_2}{\lambda\alpha_2-\delta}$ for $t\in[s+\Delta s]$.*

Proof. Suppose not. This implies $e^*(t)<e_{\max}$, $\phi^*(t)>\bar{\phi}_d$ from lemma 6. The following condition must hold

$$V^S(s)-V^U(s)=\frac{c}{\lambda}\text{ for all }s\in[t,t+\Delta t]$$

for effort to be optimal. Hence, we also require

$$\begin{aligned}V^S(t)-V^S(t+\Delta t)&=V^U(t)-V^U(t+\Delta t)\\ \left[\phi^*(t+\Delta t)-\phi^*(t)\right]\alpha_2&=V^U(t)-V^U(t+\Delta t)\end{aligned}$$

since this is true for all s. Recall

$$V^S(t) = \int_t^{t+\Delta t} -\delta(r-t)\tilde{d}^*(r)(1-\phi^*(r))\exp\left(-\int_t^r \tilde{d}^*(s)(1-\phi^*(s))\,ds\right)dr$$
$$+(V_0+\alpha_1+\alpha_2)\left[1-\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)\right]$$
$$+\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)\left[V^S(t+\Delta t)-\Delta t\delta\right]$$

and

$$V^S(t)-V^S(t+\Delta t) = \int_t^{t+\Delta t} -\delta(r-t)\tilde{d}^*(r)(1-\phi^*(r))\exp\left(-\int_t^r \tilde{d}^*(s)(1-\phi^*(s))\,ds\right)dr$$
$$+\left(V_0+\alpha_1+\alpha_2-V^S(t+\Delta t)\right)\left[1-\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)\right]$$
$$-\Delta t\delta\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,dsds\right)$$

Now we can write $V^U(t)$ as follows

$$V^U(t) = \int_t^{t+\Delta t} -\delta(r-t)\tilde{d}^*(r)(1-\phi^*(r))\exp\left(-\int_t^r \tilde{d}^*(s)(1-\phi^*(s))\,ds\right)dr$$
$$+(V_0+\alpha_1)\left(1-\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)\right)$$
$$+\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)\left[V^U(t+\Delta t)-\Delta t\delta\right]$$

since the agent is indifferent about which level of effort to exert we can write it out assuming $e_i(t)=0$. We can further calculate $V^U(t)-V^U(t+\Delta t)$ using $V^U(t+\Delta t)=V^S(t+\Delta t)-\frac{c}{\lambda}$

$$V^U(t)-V^U(t+\Delta t) = \int_t^{t+\Delta t} -\delta(r-t)\tilde{d}^*(r)(1-\phi^*(r))\exp\left(-\int_t^r \tilde{d}^*(s)(1-\phi^*(s))\,ds\right)dr$$
$$+\left(V_0+\alpha_1-V^S(t+\Delta t)+\frac{c}{\lambda}\right)\left[1-\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)\right]$$
$$-\Delta t\delta\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)$$

$$V^U(t)-V^U(t+\Delta t)=V^S(t)-V^S(t+\Delta t)+\left(\frac{c}{\lambda}-\alpha_2\right)\left[1-\exp\left(-\int_t^{t+\Delta t}\tilde{d}^*(s)(1-\phi^*(s))\,ds\right)\right]$$

Hence, we have a contradiction $V^U(t)-V^U(t+\Delta t)>V^S(t)-V^S(t+\Delta t)$ for $\tilde{d}^*(s)>0$ and $\phi^*(s)<1$. □

Lemma 8. $\phi^*(t)\geq\max\left\{\bar{\phi}_d,\bar{\phi}\right\}$

Proof. Suppose $\exists\breve{t}:\phi^*(\breve{t})<\bar{\phi}_d$ then $\exists\Delta\breve{t}>0:\phi^*(s)<\bar{\phi}_d$ for $s\in\left[\breve{t}-\Delta\breve{t},\breve{t}\right]$. However, this is a contradiction as a successful player will strictly prefer to implement for all $s\in\left[\breve{t}-\Delta\breve{t},\breve{t}\right]$ and hence $\phi^*(\breve{t})>\bar{\phi}_d$ which is a contradiction that $\exists\breve{t}:\phi^*(\breve{t})<\bar{\phi}_d$. Suppose $\exists\breve{t}:\phi^*(\breve{t})<\bar{\phi}$ then $\exists\Delta\breve{t}>0:\phi^*(\breve{t}-\Delta\breve{t})<\bar{\phi}$. This implies an upper bound on the value of information is $\left(1-\phi^*(\breve{t})\right)\alpha_2+\phi^*(\breve{t})\alpha_1$. Thus, for any time s such that $\phi^*(s)<\bar{\phi}\Rightarrow e^*(s)=0$ therefore $\frac{d\phi^*(s)}{ds}=0$ so $\lim_{r\to}\phi^*(\breve{t}^-)>\bar{\phi}(\breve{t})$ which is a violation of the continuity of

$\phi^* (t)$. This is a contradiction given the upper bound on the arrival rate of information. □

Lemma 9. *Suppose* $e^* (t) < e_{\max}$ *for some* t *then under large incentives* $\phi^* (T) = \bar{\phi}_d$ *and under small incentives* $\phi^* (T) = \bar{\phi}$

Proof. Suppose not, then $\phi^* (T) > \max \{\bar{\phi}_d, \bar{\phi}\}$. Let $\hat{t} = \inf \{t | e^* (t) < e_{\max}\}$. Being successful at time $\hat{t}$ has a continuation value given by

$$V^S (\hat{t}) = V_0 + \alpha_1 + (1 - \phi^* (T)) \alpha_2 - \delta (T - \hat{t})$$

since the optimal strategy for a successful individual is to delay until the deadline which is due to $e^* (t) = e_{\max}$ and $\phi^* (t) > \bar{\phi}_d$ for $t \geq \hat{t}$. The continuation value for the unsuccessful individual is

$$V^U (\hat{t}) = \left(V_0 + \alpha_1 + (1 - \phi^* (T)) \alpha_2 - \frac{c}{\lambda}\right) [1 - \exp (-\lambda e_{\max} (T - \hat{t}))] + \exp (-\lambda e_{\max} (T - \hat{t})) [V_0 + (1 - \phi^* (T)) \alpha_1] - \delta (T - \hat{t})$$

Thus, incentives for effort are given by

$$V^S (\hat{t}) - V^U (\hat{t}) = \frac{c}{\lambda} + \exp (-\lambda e_{\max} (T - \hat{t})) \left[(1 - \phi^* (T)) \alpha_2 + \phi^* (T) \alpha_1 - \frac{c}{\lambda}\right].$$

Further, by definition of $\bar{\phi}_d$ and $\bar{\phi}$.we have

$$(1 - \phi^* (T)) \alpha_2 + \phi^* (T) \alpha_1 > \left((1 - \bar{\phi}_d) \alpha_2 + \bar{\phi}_d \alpha_1\right) > \left((1 - \bar{\phi}) \alpha_2 + \bar{\phi} \alpha_1\right) = \frac{c}{\lambda}$$

Hence, $V^S (\hat{t}) - V^U (\hat{t}) > \frac{c}{\lambda}$ and by the continuity of V^S and V^U $\exists \omega > 0 : e^* (\hat{t} - \omega) = e_{\max}$ which is a contradiction of $\hat{t} = \inf \{t | e^* (t) < e_{\max}\}$. □

The previous lemmas restrict the set of potential equilibria to those where $\phi^* (t)$ is continuous, decreasing and bounded below by $\max \{\bar{\phi}_d, \bar{\phi}\}$. Furthermore, if implementation decisions are taken prior to the deadline, then $\frac{d\phi^*}{dt} = 0$ and either $\phi^* = 1$ or $\phi^* = \bar{\phi}_d$ during those times.

B.3.2 Proof for Uniqueness of Symmetric Equilibria Set for Large Incentives Case

Define $V^{S*} (t)$ and $V^{U*} (t)$

$$\begin{aligned} V^{S*} (t) &= V_0 + \alpha_1 + (1 - \bar{\phi}_d) \alpha_2 \\ V^{U*} (t) &= V_0 + \alpha_1 + \frac{(1 - \bar{\phi}_d) \alpha_2}{2} - \left(c + \frac{\delta}{e_{\max}}\right) \frac{1}{2\lambda} \\ &\quad - \exp (-2\lambda e_{\max} (T - t)) \left[\bar{\phi}_d \alpha_1 + \frac{(1 - \bar{\phi}_d) \alpha_2}{2} - \left(c + \frac{\delta}{e_{\max}}\right) \frac{1}{2\lambda}\right] \end{aligned}$$

Note further that

$$V^{S*} (t) - V^{U*} (t) = \frac{(1 - \bar{\phi}_d) \alpha_2}{2} + \left(c + \frac{\delta}{e_{\max}}\right) \frac{1}{2\lambda} + e^{-2\lambda(T-t)} \left[\bar{\phi}_d \alpha_1 + \frac{(1 - \bar{\phi}_d) \alpha_2}{2} - \left(c + \frac{\delta}{e_{\max}}\right) \frac{1}{2\lambda}\right]$$

and

$$V^{S*} (t) - V^{U*} (t) \begin{cases} > \frac{c}{\lambda} \text{ for } T - t < Z \\ = \frac{c}{\lambda} \text{ for } T - t = Z \\ < \frac{c}{\lambda} \text{ for } T - t > Z \end{cases}$$

Also, define $\bar{t}_d(\phi)$, $V^{S*}(t,\phi)$ and $V^{U*}(t,\phi)$

$$\bar{t}_d(\phi) = \frac{1}{\lambda}\ln\frac{\phi}{\bar{\phi}_d}$$

$$V^{S*}(t,\phi) = \begin{cases} V_0 + \alpha_1 + (1-\phi(t)\exp(-\lambda e_{\max}(T-t)))\alpha_2 - \delta(T-t) & \text{for } T-\bar{t}_d(\phi) < t \le T \\ V^{S*}(t+\tilde{t}) - \delta\tilde{t} & \text{for } T-\bar{t}_d(\phi) - Z \le t \le T-\bar{t}_d(\phi), \end{cases}$$

$$V^{U*}(t,\phi) = \begin{cases} [1-\exp(-\lambda e_{\max}(T-t))]\left(V_0+\alpha_1+(1-\phi(t)\exp(-\lambda e_{\max}(T-t)))\alpha_2 - \frac{c}{\lambda}\right) \\ \quad + \exp(-\lambda e_{\max}(T-t))(V_0 + [1-\phi(t)\exp(-\lambda e_{\max}(T-t))]\alpha_1) - \delta(T-t) \\ \qquad \text{for } T-\bar{t}_d(\phi) < t \le T \\ \left(1-\exp\left(-\lambda e_{\max}\tilde{t}\right)\right)\left(V^{S*}\left(t+\tilde{t}\right) - \frac{c}{\lambda}\right) + \exp\left(-\lambda e_{\max}\tilde{t}\right)V^{U*}\left(t+\tilde{t}\right) - \delta\tilde{t} \\ \qquad \text{for } T-\bar{t}_d(\phi) - Z \le t \le T-\bar{t}_d(\phi). \end{cases}$$

Note that

$$V^{S*}(t,\phi) - V^{U*}(t,\phi) = \begin{cases} \frac{c}{\lambda} + \exp(-\lambda e_{\max}(T-t))[\phi(t)\exp(-\lambda e_{\max}(T-t))\alpha_1 \\ \quad + (1-\phi(t)\exp(-\lambda e_{\max}(T-t)))\alpha_2 - \frac{c}{\lambda}] \\ \qquad \text{for } T-\bar{t}_d(\phi) < t \le T \\ \frac{c}{\lambda} + \exp\left(-\lambda e_{\max}\tilde{t}\right)\left(V^{S*}\left(t+\tilde{t}\right) - V^{U*}\left(t+\tilde{t}\right) - \frac{c}{\lambda}\right) \\ \qquad \text{for } T-\bar{t}_d(\phi) - Z \le t \le T-\bar{t}_d(\phi). \end{cases}$$

Hence,

$$V^{S*}(t,\phi) - V^{U*}(t,\phi) \begin{cases} > \frac{c}{\lambda} & \text{for } T-\bar{t}_d(\phi) - Z < t \\ = \frac{c}{\lambda} & \text{for } T-\bar{t}_d(\phi) - Z = t \\ < \frac{c}{\lambda} & \text{for } T-\bar{t}_d(\phi) - Z > t \end{cases}.$$

Lemma 10. *The unique equilibrium strategies in any subgame starting at t with beliefs $\phi(t) \ge \bar{\phi}_d$ such that $t \ge T - Z - \bar{t}_d(\phi)$ are $e^*(s) = e_{\max}$ for $t \le s \le T$ and* $d^*(s) = \begin{cases} 0 & \text{for } t \le s \le \bar{t}_d(\phi) \\ \frac{\lambda^2\alpha_2}{\lambda\alpha_2 - \delta} & \text{for } \bar{t}_d(\phi) < s \le T \end{cases}$

Proof. Suppose $\exists s \ge t, \varepsilon > 0$ such that $e^*(r) < e_{\max}$ for $r \in [s-\varepsilon, s)$. If this is the case, we can check the continuation values at $\widehat{s}$ where

$$\widehat{s} = \sup\{r | e^*(r) < 1\}.$$

Given that $e^*(s) = e_{\max}$ for $r \ge \widehat{s}$ then the unique implementation decision strategy is

$$d^*(s) = \begin{cases} 0 & \text{for } \widehat{s} \le s \le \min\{T, \widehat{s} + \bar{t}_d(\phi(\widehat{s}))\} \\ \frac{\lambda^2\alpha_2}{\lambda\alpha_2 - \delta} & \text{for } \widehat{s} + \bar{t}_d(\phi(\widehat{s})) < s \le T \text{ if } \widehat{s} + \tilde{t}(\phi(\widehat{s})) < T \end{cases}$$

since the only belief at which a successful individual will implement is $\phi = \bar{\phi}_d$ when the unsuccessful agent is exerting maximum effort. We can therefore write the continuation values as $V^{S*}(\widehat{s},\phi)$ and $V^{U*}(\widehat{s},\phi)$. The contradiction now comes from noting that for $t > T - Z - \bar{t}_d(\phi)$ and $V^{S*}(t,\phi) - V^{U*}(t,\phi) > \frac{c}{\lambda}$. Therefore, $\exists\zeta : V^{S*}(r,\phi(r)) - V^{U*}(r,\phi(r)) > \frac{c}{\lambda}$ and $e^*(r) < e_{\max}$ for $r \in [\widehat{s}-\zeta, \widehat{s})$ which means $e^*(r)$ is not an equilibrium strategy. ☐

Lemma 11. *Suppose $T \ge X_d + Z$, then an upper bound on $\phi^*(t)$ is given by*

$$\phi^*(t) \le \begin{matrix} \bar{\phi}_d \exp(\lambda(T-Z-t)) & \text{for } T - X_d - Z \le t < T - Z \\ \bar{\phi}_d & \text{for } T - Z \le t \le T \end{matrix}$$

Proof. Suppose $\exists t', \phi^*(t') > \bar{\phi}_d \exp(\lambda(T-Z-t'))$ for $T - X_d - Z \le t' < T - Z$ or $\phi^*(t) > \bar{\phi}_d$ for $T - Z \le t' \le T$ then

$$\exists s < t' : (s, \phi^*(s)) \in \left\{ \begin{matrix} (r(\phi),\phi) | r(\phi) = T - X_d - Z + \frac{1}{\lambda e_{\max}}\ln\frac{1}{\phi} \\ +\gamma\left(t' - \left(T - X_d - Z + \frac{1}{\lambda e_{\max}}\ln\frac{1}{\phi}\right)\right), \gamma \in (0,1), \phi \in [\phi(t), 1] \end{matrix} \right\}.$$

Now $s > T - Z - \bar{t}_d(\phi^*(s))$ and thus the unique equilibrium of the subgame starting from $(s, \phi^*(s))$ is given by Lemma 10. However, the Bayesian belief $\widehat{\phi}^*(r)$ in this subgame reaches

$$\widehat{\phi}^*(r) = \phi^*(t') = (T - X_d - Z) + \frac{1}{\lambda e_{\max}} \ln \frac{1}{\phi^*(t')} + \gamma(s)\left(t' - (T - X_d - Z) + \frac{1}{\lambda e_{\max}} \ln \frac{1}{\phi^*(t')}\right)$$

where $\gamma(s) = \frac{s-(T-X_d-Z)+\frac{1}{\lambda e_{\max}}\ln\frac{1}{\phi^*(s)}}{t'-(T-X_d-Z)+\frac{1}{\lambda e_{\max}}\ln\frac{1}{\phi^*(t')}} < 1$. Thus, $r < t$ and $\widehat{\phi}^*(t') > \phi^*(t')$ and hence $\phi^*(t')$ is not part of a perfect Bayesian equilibrium. □

Together with lemma 8 this uniquely determines $\phi(t) = \bar{\phi}_d$ for $t \geq T - Z$ if $T \geq X_d + Z$.

Lemma 12. *Suppose $T \geq X_d + Z$ then a lower bound on $\phi^*(t)$ is given by*

$$\phi^*(t) \geq \begin{cases} \bar{\phi}_d \text{ for } t \geq T - Z \\ \bar{\phi}_d + \delta(T - Z - t) \text{ for } T - Y_d - Z \leq t < T - Z \\ 1 \text{ for } t \leq T - Y_d - Z \end{cases}$$

Proof. Lemma 11 and lemma 8 pin down $\phi^*(t) = \bar{\phi}_d$ for $t \geq T - Z$. Now suppose $\exists s < T - Y_d - Z : \phi^*(s) < 1$ or $\exists s : T - Y_d - Z \leq s < T - Z, \phi^*(s) < \bar{\phi}_d + \delta(T - Z - s)$. If $\exists r < T - Z : d^*(t) = 0$ for $t \in [r, T - Z]$ there is an immediate contradiction as successful individuals would strictly prefer to implement the project immediately. If not then using lemma 7 if $d^*(t) > 0$ for $s \leq t < T - Z$ then $e^*(t) = e_{\max}, \phi^*(t) = \bar{\phi}_d, d^*(t) = \frac{\lambda^2\alpha_2}{\lambda\alpha_2 - \delta}$ for $t \in [r, T - Z]$. However in this case we also have a contradiction as $V^S(t) - V^U(t) = V^{S*}(t) - V^{U*}(t) < \frac{c}{\lambda}$ since $t < T - Z$ and the effort strategy $e^*(t) = e_{\max}$ is not optimal and cannot be part of an equilibrium. □

These two lemmas provide an upper and lower bound on the values of $\phi^*(t)$ in equilibrium. The proof for uniqueness now proceeds by showing that the only equilibrium strategies which support values of ϕ between these bounds are the ones given in the propositions.

Proof. **i) Case 1:** $T < X_d$

Strategy of successful player: $d^*(t) = 0$ for all t. Strategy of unsuccessful player: $e^*(t) = e_{\max}$ for all t. Beliefs: $\phi^*(t) = \exp(-\lambda e_{\max} t)$ for all t.

ii) Case 2: $X_d \leq T < X_d + Z$

Strategy of successful player: $d^*(t) = 0$ for $t < X_d$, $d^*(t) = \frac{\lambda^2\alpha_2}{\lambda\alpha_2 - \delta}$ for $t \geq X_d$. Strategy of unsuccessful player: $e^*(t) = e_{\max}$ for all t. Beliefs: $\phi^*(t) = \exp(-\lambda e_{\max} t)$ for $t < X_d$.and $\phi(t) = \bar{\phi}_d$ for $t \geq X_d$.

Lemma 10 covers Case 1 and 2.

iii) Case 3: $X_d + Z < T < Y_d + Z$

Strategy of successful player: $d^*(t) = 0$ for $t < T - Z$ and $d^*(t) = \frac{\lambda^2\alpha_2}{\lambda\alpha_2 - \delta}$ for $t \geq T - Z$. Strategy of unsuccessful player: $d^*(t) = 0$ for all t and $e^*(t)$ satisfies

$$\exp\left(-\lambda \int_0^t e^*(s)\, ds\right) \geq \bar{\phi}_d + (T - Z - t)\frac{\delta}{\alpha_2}$$

and

$$\exp\left(-\lambda \int_0^{T-Z} e^*(s)\, ds\right) = \bar{\phi}_d.$$

for $t < T - Z$ and $e^*(t) = e_{\max}$ for all $t \geq T - Z$. Beliefs: $\phi^*(t) = \exp\left(-\lambda \int_0^t e^*(s)\, ds\right)$ for $t < T - Z$ and $\phi^*(t) = \bar{\phi}_d$ for $t \geq T - Z$.

Lemmas 10, 12 and 11 determine the bounds on $\phi^*(t)$ and that $\phi^*(t) = \bar{\phi}_d, e^*(t) = e_{\max}$ for $t \geq T - Z$. Lemma 7 implies that $d^*(t) = 0$ for $t < T - Z$ which suffices along with the earlier lemmas for the equilibrium strategy set.

iv) Case 4: $T > Y_d + Z$

Strategy of successful player: $d^*(t) = \textit{implement}$ for $0 \le t \le T - Y_d - Z$ and $d^*(t) = 0$ for $T - Y_d - Z < t < T - Z$ and $d^*(t) = \frac{\lambda^2\alpha_2}{\lambda\alpha_2 - \delta}$ for $t \ge T - Z$. Strategy of unsuccessful player: $d^*(t) = 0$ for all t. $e^*(t) = \frac{\delta}{c}$ for $0 \le t \le T - Y_d - Z$ and $e^*(t)$ satisfies

$$\exp\left(-\lambda \int_{T-Y_d-Z}^{t} e^*(s)\, ds\right) \ge \bar{\phi}_d + (T - Z - t)\frac{\delta}{\alpha_2}$$

and

$$\exp\left(-\lambda \int_{T-Y_d-Z}^{T-Z} e^*(s)\, ds\right) = \bar{\phi}_d.$$

for $T - Y_d - Z < t < T - Z$ and $e^*(t) = e_{\max}$ for $t \ge T - Z$.

Beliefs: $\phi^*(t) = 1$ for $0 \le t \le T - Y_d - Z$, $\phi^*(t) = \exp\left(-\lambda \int_{T-Y_d-Z}^{t} e^*(s)\, ds\right)$ for $T - Y_d - Z < t < T - Z$ and $\phi^*(t) = \bar{\phi}_d$ for $t \ge T - Z$.

Case 3 above covers the subgames for $t > T - Z - \widehat{T}$. It remains to show that the above strategies are unique for $t \le T - Z - \widehat{T}$. For $t \le T - Z - \widehat{T}$ we have shown that $\phi^*(t) = 1$. First, rule out that $e(t) = 0$. If this were the case the continuation payoffs would be $V^S(t) = V_0 + \alpha_1$ and $V^U(t) = V_0 + \alpha_1 - \frac{c}{\lambda} - (\tilde{t} - t)\delta$ where $\tilde{t} = \inf\{s > t : e(s) > 0\}$, therefore the strategy $e^*(t) = 0$ is not optimal as $V^S - V^U > \frac{c}{\lambda}$. Implying that $0 < e^*(t) \le \frac{\delta}{\lambda\alpha_2}$ and individuals decide to implement immediately. We have $V^S(t) = V_0 + \alpha_1$ so $V^U(t) = V_0 + \alpha_1 - \frac{c}{\lambda}$ for $t \le T - Z - \widehat{T}$.

$$V^U(t) = \int_t^{t+\Delta t} \left(V_0 + \alpha_1 - c\int_t^s e^*(r)\, dr - \delta(s-t)\right) 2\lambda e^*(s) \exp\left(-2\lambda \int_t^s e^*(r)\, dr\right) ds$$
$$+ \exp\left(-2\lambda \int_t^{t+\Delta t} e^*(r)\, dr\right)\left(V^U(t+\Delta t) - \delta\Delta t - c\int_t^{t+\Delta t} e^*(r)\, dr\right)$$

$$V^U(t) = \left(V_0 + \alpha_1 - \frac{c}{2\lambda}\right)\left(1 - \exp\left(-2\lambda \int_t^{t+\Delta t} e^*(r)\, dr\right)\right)$$
$$- \int_t^{t+\Delta t} \delta(s-t)\, 2\lambda e^*(s) \exp\left(-2\lambda \int_t^s e^*(r)\, dr\right) ds$$
$$+ \exp\left(-2\lambda \int_t^{t+\Delta t} e^*(r)\, dr\right)\left(V^U(t+\Delta t) - \delta\Delta t\right)$$

$$V^U(t) = V_0 + \alpha_1 - \frac{c}{\lambda} + \frac{c}{2\lambda}\left(1 - \exp\left(-2\lambda \int_t^{t+\Delta t} e^*(r)\, dr\right)\right)$$
$$- \int_t^{t+\Delta t} \delta(s-t)\, 2\lambda e^*(s) \exp\left(-2\lambda \int_t^s e^*(r)\, dr\right) ds$$
$$- \exp\left(-2\lambda \int_t^{t+\Delta t} e^*(r)\, dr\right)\delta\Delta t$$

hence

$$\frac{c}{2\lambda}\left(1 - \exp\left(-2\lambda \int_t^{t+\Delta t} e^*(r)\, dr\right)\right) = \int_t^{t+\Delta t} \delta(s-t)\, 2\lambda e^*(s) \exp\left(-2\lambda \int_t^s e^*(r)\, dr\right) ds$$
$$+ \exp\left(-2\lambda \int_t^{t+\Delta t} e^*(r)\, dr\right)\delta\Delta t$$

$$ce^*(t)\,\Delta t + O\left(\Delta t^2\right) = \delta \Delta t + O\left(\Delta t^2\right)$$

We therefore require that $e^*(t) = \frac{\delta}{c}$. □

B.3.3 Proof for Uniqueness of Symmetric Equilibria Set for Baseline Case

Define

$$\begin{aligned} \bar{t}_e(\phi) &= \frac{1}{\lambda}\ln\frac{\phi}{\bar{\phi}} \\ V_e^{S*}(t,\phi) &= V_0 + \alpha_1 + (1-\phi\exp[-\lambda e_{\max}(T-t)])\,\alpha_2 - \delta \bar{t}_e(\phi) \end{aligned}$$

as well as

$$\begin{aligned} V_e^{U*}(t,\phi) = (1-\exp[-\lambda e_{\max}(T-t)])\left(V_0 + \alpha_1 + (1-\phi\exp[-\lambda e_{\max}(T-t)])\,\alpha_2 - \frac{c}{\lambda}\right) \\ + \exp[-\lambda e_{\max}(T-t)]\left(V_0 + (1-\phi\exp[-\lambda e_{\max}(T-t)])\,\alpha_1\right) - \delta \bar{t}_e(\phi) \end{aligned}$$

and

$$\begin{aligned} V_e^{S*}(t,\phi) - V_e^{U*}(t,\phi) = \frac{c}{\lambda} \\ + \exp[-\lambda e_{\max}(T-t)]\left(\phi\exp[-\lambda e_{\max}(T-t)]\,\alpha_1 + (1-\phi\exp[-\lambda e_{\max}(T-t)])\,\alpha_2 - \frac{c}{\lambda}\right) \end{aligned}$$

Note that

$$V_e^{S*}(t,\phi) - V_e^{U*}(t,\phi) > \frac{c}{\lambda}$$

provided that $t < \bar{t}_e(\phi)$.

Lemma 13. *The unique equilibrium strategy in any subgame starting at t with beliefs $\phi(t)$ such that $t \geq T - \bar{t}_e(\phi)$ is $e^*(s) = e_{\max}$ for $t \leq s \leq T$ and $d^*(s) = 0$.*

Proof. Suppose $\exists s \geq t, \varepsilon > 0$ such that $e^*(r) < e_{\max}$ for $r \in [s-\varepsilon, s)$. If this is the case, we can check the continuation values at $\widehat{s}$ where

$$\widehat{s} = \sup\{r | e^*(r) < 1\}.$$

Given that $e^*(r) = e_{\max}$ for $r \geq \widehat{s}$ then the unique decision strategy is $d^*(r) = 0$ since the only belief at which a successful individual will decide to implement is $\bar{\phi}_d < \bar{\phi}$ when the unsuccessful agent is exerting maximum effort. We can therefore write the continuation values as $V_e^{S*}(\widehat{s}, \phi(\widehat{s}))$, $V_e^{U*}(\widehat{s}, \phi(\widehat{s}))$. The contradiction now comes from noting that

$$t > T - \bar{t}_e(\phi) \Rightarrow \widehat{s} > T - \bar{t}_e(\phi(\widehat{s}))$$

hence $V_e^{S*}(\widehat{s}, \phi(\widehat{s})) - V_e^{U*}(\widehat{s}, \phi(\widehat{s})) > \frac{c}{\lambda}$. Thus $\exists \zeta : V_e^{S*}(r, \phi(r)) - V_e^{U*}(r, \phi(r)) > \frac{c}{\lambda}$ and $e^*(r) < e_{\max}$ for $r \in [\widehat{s} - \zeta, \widehat{s})$ which means that $e^*(r)$ is not an equilibrium strategy. □

Lemma 14. *Suppose $T \geq X$, then an upper bound on $\phi^*(t)$ is given by*

$$\phi^*(t) \leq \begin{cases} 1 \text{ for } \; t < T - X \\ \bar{\phi}\exp(\lambda(T-t)) \; \text{ for } \; T - X \leq t < T \end{cases}$$

Proof. Suppose $\exists t'$ such that $\phi^*(t') > \bar{\phi}\exp(\lambda(T-t))$ for $T - X \leq t' < T$ then

$$\exists (s, \phi^*(s)) \in \left\{ \begin{array}{c} (r(\phi), \phi) \,|\, r(\phi) = T - X + \frac{1}{\lambda e_{\max}}\ln\frac{1}{\phi} \\ +\gamma\left(t' - \left(T - X + \frac{1}{\lambda e_{\max}}\ln\frac{1}{\phi}\right)\right), \gamma \in (0,1), \phi \in [\phi(t'), 1] \end{array} \right\}$$

Now $s > T - \bar{t}_e(\phi^*(s))$, so the unique equilibrium of the subgame starting from $(s, \phi^*(s))$ is given by Lemma 14. However, the Bayesian belief $\widehat{\phi}^*(r)$ in this subgame reaches $\widehat{\phi}^*(r) = \phi^*(t')$ at $r = (T - X) + \frac{1}{\lambda e_{\max}}\ln\frac{1}{\phi^*(t')} +$

$\gamma(s)\left[t' - (T-X) + \frac{1}{\lambda e_{\max}} \ln \frac{1}{\phi^*(t')}\right]$ where

$$\gamma(s) = \frac{s - (T-X) + \frac{1}{\lambda e_{\max}} \ln \frac{1}{\phi^*(s)}}{t' - (T-X) + \frac{1}{\lambda e_{\max}} \ln \frac{1}{\phi^*(t')}} < 1$$

Thus, $r < t'$ and $\widehat{\phi}^*(t') > \phi^*(t')$ and hence $\phi^*(t')$ is not part of the perfect Bayesian equilibrium. □

This uniquely determines $\phi^*(T) = \bar{\phi}$ for $T \geq X$.

Lemma 15. *Suppose $T \geq X$ then a lower bound on $\phi^*(t)$ is given by*

$$\phi^*(t) \geq \begin{cases} \bar{\phi} + \delta(T-t) \text{ for } T - Y \leq t < T \\ 1 \text{ for } t \leq T - Y \end{cases}$$

Proof. As noted above $\phi^*(T) = \bar{\phi}$, if $T \geq X$. Now suppose $\exists s : \phi^*(s) < 1$ for $s < T - Y$ or $\phi^*(s) < \bar{\phi} + \delta(T-s)$ for $T - Y \leq t < T$. If this is the case then using lemma 7 $d^*(t) = 0$ for $t \in [s, T]$ and there is an immediate contradiction as successful individuals will strictly prefer to implement at s than wait until T. □

These lemmas provide an upper and lower bound on the values of $\phi^*(t)$ in equilibrium. As before, the proof for uniqueness now proceeds by showing that the only equilibrium strategies which support values of ϕ between these bounds are the ones given in the propositions.

Proof. **i) Case 1:** $T < X$

Strategy of successful player: $d^*(t) = 0$ for all t. Strategy of unsuccessful player: $d^*(t) = 0$ for all t and $e^*(t) = e_{\max}$ for all t. Beliefs: Beliefs evolve according to $\phi^*(t) = \exp\left(-\lambda \int_0^t e^*(s)\,ds\right)$ for all t.

This follows immediately from Lemma 13.

ii) Case 2: $X < T < Y$

Strategy of successful player: $d^*(t) = 0$ for all t. Strategy of unsuccessful player: $d^*(t) = 0$ for all t and $e^*(t)$ satisfies

$$\exp\left(-\lambda \int_0^t e^*(s)\,ds\right) \geq \bar{\phi} + (T-t)\frac{\delta}{\alpha_2}$$

and

$$\exp\left(-\lambda \int_0^T e^*(s)\,ds\right) = \bar{\phi}.$$

Beliefs: Beliefs evolve according to $\phi^*(t) = \exp\left(-\lambda \int_0^t e^*(s)\,ds\right)$ for all t.

Lemma 7 and $\phi^*(T) = \bar{\phi}$ (as shown above from Lemmas 13 and 14) imply that $d^*(t) = 0$ for all t. The restriction on $e^*(t)$ comes from Lemma 15. Beliefs are given by Bayesian updating.

iii) Case 3: $T > \frac{1}{\delta}(1-\bar{\phi})\alpha_2$

Strategy of successful player: $d^*(t) = \textit{implement}$ for $t < T - Y$ and $d(t) = 0$ for $t \geq T - Y$

Strategy of unsuccessful player: $d^*(t) = 0$ for all t and $e^*(t) = \frac{\delta}{c}$ for $t < T - Y$. $e(t)$ satisfies

$$\exp\left(-\lambda \int_{T-Y}^t e^*(s)\,ds\right) \geq \bar{\phi} + (T-t)\frac{\delta}{\alpha_2}$$

and

$$\exp\left(-\lambda \int_{T-Y}^T e^*(s)\,ds\right) = \bar{\phi}.$$

for $t \geq T - Y$. Beliefs: $\phi(t) = 1$ for $t < T - Y$ and $\phi^*(t) = \exp\left(-\lambda \int_0^t e^*(s)\,ds\right)$ for $t \geq T - Y$.

The proof for Case 2 encompasses the subgames for $t \geq T - Y$. The uniqueness for $t < T - Y$ is completely analogous to the proof in case 4 for large incentives of the uniqueness of equilibrium strategies for $t < T - Y_d - Z$. □

B.4 Modes of Communication

In this section, we provide an explicit characterization of the setup and the results when relaxing our assumptions about the value of breakthroughs and the modes of communication as discussed in Section 6.2.

B.4.1 Setup

We assume that an individual is prepared to exert effort for a second breakthrough, $\alpha_2 > \frac{c}{\lambda} + \frac{\delta}{\lambda e_{\max}}$. We also assume that an individual is prepared to delay implementation if the other player is exerting maximum effort to produce a third breakthrough, $\lambda \alpha_3 e_{\max} > \delta$, and that this is no longer true for a fourth breakthrough, $\lambda \alpha_4 e_{\max} < \delta$.

From a modeling standpoint it is also reasonable to draw a distinction between implementing the project and communicating that a breakthrough has been produced. Hence we allow agents to reveal at each instance that they have produced a breakthrough without implementing the project. We denote this action by $w_i(t, n_i) : [0, T] \times \{0, 1, 2\} \to \{0, 1, ..., n_i\}$ where n_i denotes the breakthrough an agent has produced. We restrict this to be non-decreasing such that an agent may only reveal that a breakthrough was produced but cannot subsequently conceal this having previously revealed it. To simplify notation we also assume that if the other agent has revealed a breakthrough, $w_{-i} = 1$ then the action w_i is no longer available. Common knowledge about the production of 2 breakthroughs will lead to a decision in a similar way to common knowledge of the production of a single breakthrough in the earlier model would also lead to a decision. Hence in the event that the other agent has previously announced the production of a breakthrough then announcing and deciding to implement are equivalent for an agent.

The history of the game at a given point in time is given by (μ, ν). $\mu \in \{0, 1, 2\}$ indicates whether no breakthrough has been announced, $\mu = 0$, or if one has been announced which player announced it, $\mu = 1, 2$. $\nu \in [0, t]$ indicates at what time the player announced successful production of a breakthrough $\nu \in (0, t]$. If no breakthrough has been announced, $\nu = 0$.

B.4.2 Short Deadline

We first consider a short deadline. The equilibrium exhibits no revelation of private information about successful breakthroughs, $w_i(t, n_i) = 0$, no decisions on the equilibrium path prior to the deadline, maximum effort by agents who have only produced one or two breakthroughs and zero effort by those who have produced two breakthroughs.

We define $\tilde{X}$ as

$$\lambda \left[\begin{array}{l} \exp(-\lambda e_{\max} \tilde{X}) \alpha_2 + \lambda e_{\max} \tilde{X} \exp(-\lambda e_{\max} \tilde{X}) \alpha_3 \\ + \left(1 - \exp(-\lambda e_{\max} \tilde{X}) - \lambda e_{\max} \tilde{X} \exp(-\lambda e_{\max} \tilde{X})\right) \alpha_4 \end{array} \right] = c.$$

Note that

$$\exp(-\lambda e_{\max} \tilde{X}) \alpha_2 + \left(1 - \exp(-\lambda e_{\max} \tilde{X})\right) \alpha_3 > \frac{c}{\lambda}.$$

Also, we assume parameter values such that $\tilde{X} < \tilde{X}_d$, where $\tilde{X}_d$ is given by

$$\lambda e_{\max} \left[\exp(-\lambda e_{\max} \tilde{X}_d) \alpha_3 + \lambda e_{\max} \tilde{X}_d \exp(-\lambda e_{\max} \tilde{X}_d) \alpha_4 \right] = \delta.$$

Proposition 9. *$\exists T > 0$ such that a symmetric Perfect Bayesian Equilibrium is*

$$
\begin{aligned}
w_i^*(t, n_i) &= 0 \\
d_i^*(t, n_i, \mu, \nu) &= \begin{cases} \textit{implement if } \mu \in \{1,2\} \\ 0 \textit{ otherwise} \end{cases} \\
e_i^*(t, n_i, \mu, \nu) &= \begin{cases} 0 \textit{ if } \mu \in \{1,2\} \textit{ or } n_i \geq 2 \\ e_{\max} \textit{ otherwise} \end{cases} \\
\phi_i^*(t, \mu, \nu) &= \begin{cases} \{0,0,1\} \textit{ if } \mu \in \{1,2\} \\ \{\exp(-\lambda e_{\max} t), \lambda e_{\max} t \exp(-\lambda e_{\max} t), 1 - (\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t)\} \textit{ otherwise.} \end{cases}
\end{aligned}
$$

Proof. Define $V_i(t, n_i, \mu, \nu)$ as the continuation value for an agent at time t depending on the breakthrough they have produced and the history of announcements of breakthroughs by themselves and the other agent.
Define $\chi(n_i)$ as

$$\chi(n_i) = V_0 + \sum_{j=1}^{n_i} \alpha_j + (1 - \exp(-\lambda e_{\max} T))\, \alpha_{n_i+1} + (1 - \exp(-\lambda e_{\max} T) - \lambda e_{\max} T \exp(-\lambda e_{\max} T))\, \alpha_{n_i+2}.$$

This is the expected value of the project at the deadline conditional on an agent having n_i breakthroughs in equilibrium. Note that for $n_i = 1, 2$ and $T < \tilde{X}$,

$$
\begin{aligned}
\chi(n_i) - \chi(n_i - 1) &= \exp(-\lambda e_{\max} T)\alpha_{n_i} + \lambda e_{\max} T \exp(-\lambda e_{\max} T)\alpha_{n_i+1} \\
&\quad + (1 - \exp(-\lambda e_{\max} T) - \lambda e_{\max} T \exp(-\lambda e_{\max} T))\, \alpha_{n_i+2} \\
&> \frac{c}{\lambda}.
\end{aligned}
$$

The potential histories in the game can be organized into the following three categories:

$$
\begin{aligned}
(1) &: (t,0,0,0);(t,1,0,0);(t,2,0,0) \\
(2) &: (t,0,-i,v);(t,1,-i,v);(t,2,-i,v) \\
(3) &: (t,1,i,\nu);(t,2,i,\nu).
\end{aligned}
$$

The first category contains the histories on the equilibrium path. The continuation values for the first category are:

$$
\begin{aligned}
V_i(t,2,0,0) &= \chi(2) - \delta(T-t) \\
V_i(t,1,0,0) &= (1 - \exp(-\lambda e_{\max}(T-t)))\,\chi(2) + \exp(-\lambda e_{\max}(T-t))\chi(1) \\
&\quad - \frac{c}{\lambda}(1 - \exp(-\lambda e_{\max}(T-t))) - \delta(T-t) \\
V_i(t,0,0,0) &= \exp(-\lambda e_{\max}(T-t))\chi(0) + \lambda e_{\max}(T-t)\exp(-\lambda e_{\max}(T-t))\chi(1) \\
&\quad + (1 - \exp(-\lambda e_{\max}(T-t)) - \lambda e_{\max}(T-t)\exp(-\lambda e_{\max}(T-t)))\,\chi(2) \\
&\quad - \delta(T-t) - \frac{2c}{\lambda}\left(1 - \exp[-\lambda e_{\max}(T-t)] - \frac{\lambda e_{\max}(T-t)}{2}\exp[-\lambda e_{\max}(T-t)]\right).
\end{aligned}
$$

The off-equilibrium-path continuation values for the second category are given by:

$$
\begin{aligned}
V_i(t,0,-i,v) &= V_0 + \alpha_1 + \alpha_2 \\
V_i(t,1,-i,v) &= V_0 + \alpha_1 + \alpha_2 + \alpha_3 \\
V_i(t,2,-i,v) &= V_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4
\end{aligned}
$$

The off-equilibrium-path continuation values for the third category are given by:

$$V_i(t,1,i,v) = \begin{cases} V_0 + \alpha_1 + (1-\exp(-\lambda e_{\max} t))\,\alpha_2 + (1-(\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t))\,\alpha_3 \text{ for } \nu = t \\ V_0 + \alpha_1 + \alpha_2 + \alpha_3 \text{ for } \nu < t \end{cases}$$
$$V_i(t,2,i,v) = \begin{cases} V_0 + \alpha_1 + \alpha_2 + (1-\exp(-\lambda e_{\max} t))\,\alpha_3 + (1-(\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t))\,\alpha_4 \text{ for } \nu = t \\ V_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \text{ for } \nu < t. \end{cases}$$

Note that $\frac{c}{\lambda} > \alpha_3$ immediately implies that an effort intensity of 0 is optimal when an agent believes that at least two breakthroughs have been produced. This is the case when an agent herself has produced 2 breakthroughs or the agent is at a history where both agents have announced that they have produced a breakthrough. In the remaining cases effort is $e_{\max}$ which is optimal provided that the continuation values satisfy:

$$\begin{aligned} V_i(t,2,0,0) - V_i(t,1,0,0) &> \frac{c}{\lambda} \\ V_i(t,1,0,0) - V_i(t,0,0,0) &> \frac{c}{\lambda} \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} V_i(t,2,0,0) - V_i(t,1,0,0) &= \frac{c}{\lambda} + \exp(-\lambda e_{\max}(T-t))\left[\chi(2) - \chi(1) - \frac{c}{\lambda}\right] \\ &> \frac{c}{\lambda} \\ V_i(t,1,0,0) - V_i(t,0,0,0) &= \lambda e_{\max}(T-t)\exp(-\lambda e_{\max}(T-t))\,(\chi(2) - \chi(1)) \\ &\quad + \exp(-\lambda e_{\max}(T-t))\,(\chi(1) - \chi(0)) \\ &\quad \frac{c}{\lambda}(1 - \exp(-\lambda e_{\max}(T-t))) + c e_{\max}(T-t)\exp(-\lambda e_{\max}(T-t)) \\ &= \frac{c}{\lambda} + \lambda e_{\max}(T-t)\exp(-\lambda e_{\max}(T-t))\left(\chi(2) - \chi(1) - \frac{c}{\lambda}\right) \\ &\quad + \exp(-\lambda e_{\max}(T-t))\left(\chi(1) - \chi(0) - \frac{c}{\lambda}\right) \\ &> \frac{c}{\lambda}. \end{aligned}$$

The implementation decision strategy is optimal provided that

$$\begin{aligned} V_i(t,2,0,0) &\geq V_0 + \alpha_1 + \alpha_2 + (1-\exp(-\lambda e_{\max} t))\,\alpha_3 + (1-(\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t))\,\alpha_4 \\ V_i(t,1,0,0) &\geq V_0 + \alpha_1 + (1-\exp(-\lambda e_{\max} t))\,\alpha_2 + (1-(\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t))\,\alpha_3 \\ V_i(t,0,0,0) &\geq V_0 + (1-\exp(-\lambda e_{\max} t))\,\alpha_1 + (1-(\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t))\,\alpha_2 \end{aligned}$$

Taking the first constraint,

$$\chi(2) - \delta(T-t) \geq V_0 + \alpha_1 + \alpha_2 + (1-\exp(-\lambda e_{\max} t))\,\alpha_3 + (1-(\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t))\,\alpha_4$$

$\Leftrightarrow$

$$\begin{aligned} \delta(T-t) \leq\ & [(\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t) - (\lambda e_{\max} T + 1)\exp(-\lambda e_{\max} T)]\,\alpha_4 \\ & + \exp(-\lambda e_{\max} t)\,(1 - \exp(-\lambda e_{\max}(T-t)))\,\alpha_3. \end{aligned}$$

At $t = T$ the inequality is satisfied. The derivative of the LHS wrt t is $-\delta$ the derivative of the RHS is

$$-\lambda e_{\max}\left[\exp(-\lambda e_{\max} t)\alpha_3 + \lambda e_{\max} t \exp(-\lambda e_{\max} t)\alpha_4\right].$$

which for $0 \leq t < \tilde{X}_d$ is strictly less than $-\delta$ hence the inequality is satisfied for $0 \leq t \leq T\left(\leq \tilde{X}\right)$. We may apply the same argument for the other two constraints after noting:

$$\begin{aligned} V_i(t,1,0,0) &\geq (1-\exp(-\lambda e_{\max} T))\,\alpha_2 + (1-(\lambda e_{\max} T + 1)\exp(-\lambda e_{\max} T))\,\alpha_3 - \delta(T-t) \\ V_i(t,0,0,0) &\geq (1-\exp(-\lambda e_{\max} T))\,\alpha_1 + (1-(\lambda e_{\max} T + 1)\exp(-\lambda e_{\max} T))\,\alpha_2 - \delta(T-t). \end{aligned}$$

This implies that the announcement strategy is optimal as well,

$$\begin{aligned} V_i(t,2,0,0) &\geq V_i(t,2,i,t) \\ V_i(t,1,0,0) &\geq V_i(t,1,i,t) \end{aligned}$$

These conditions are identical to the conditions for the implementation decision strategy earlier. Finally, for the optimality of implementation decisions after an announcement, we require:

$$\begin{aligned} V_i(t,2,-i,v) &\geq V_i(s,2,-i,v) - \delta(s-t) \text{ for all } s \in (t,T] \\ V_i(t,1,-i,v) &\geq V_i(s,1,-i,v) - \delta(s-t) \text{ for all } s \in (t,T] \\ V_i(t,0,-i,v) &\geq V_i(s,0,-i,v) - \delta(s-t) \text{ for all } s \in (t,T] \\ V_i(t,2,i,v) &\geq V_i(s,2,i,v) - \delta(s-t) \text{ for all } s \in (t,T] \\ V_i(t,1,i,v) &\geq V_i(s,1,i,v) - \delta(s-t) \text{ for all } s \in (t,T] \end{aligned}$$

These are all satisfied as $V_i(t,\cdot,-i,v) = V_i(s,\cdot,-i,v)$ and $V_i(t,\cdot,i,v) = V_i(s,\cdot,i,v)$. □

Note that in the proposed equilibrium, the off-equilibrium-path belief held by a player when her partner announces that she produced some number of breakthroughs, is that her partner produced two breakthroughs. This off-equilibrium-path belief is reasonable, because a player with one breakthrough prefers not to disclose that breakthrough even if she was believed to have one breakthrough. This is true because the best response of the other player would be (i) to implement the project if she had 2 breakthroughs, (ii) to stop exerting effort and to delay if she held one breakthrough, and (iii) to continue exerting effort until she produces one breakthrough and delay if she had 0 breakthroughs. The benefit of announcing successful breakthroughs for the player with only one breakthrough is that if the other player holds 2 breakthroughs then the project is implemented immediately. The cost is that the other player would no longer exert effort for a second breakthrough after having produced a first breakthrough. It is clear that for sufficiently short deadlines, the probability that the other player has produced 2 breakthroughs is too low for the benefit of announcing successful production to outweigh the cost of reducing the effort incentives of the other player. We show this more formally below.

The expected payoff from disclosing production of one breakthrough under this scenario equals

$$\zeta \equiv (1 - (\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t))(V_0 + \alpha_1 + \alpha_2 + \alpha_3) + (\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t)\hat{V},$$

where

$$\begin{aligned} \hat{V} = {} & V_0 + \alpha_1 + \exp(-\lambda e_{\max}(T-t))\left(1 - \frac{\exp(-\lambda e_{\max} T)}{(\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t)}\right)\alpha_2 \\ & + (1 - \exp(-\lambda e_{\max}(T-t)))\left[\begin{array}{c} \frac{\exp(-\lambda e_{\max} T)}{(\lambda e_{\max} t+1)\exp(-\lambda e_{\max} t)}\alpha_2 \\ + \left(1 - \frac{\exp(-\lambda e_{\max} T)}{(\lambda e_{\max} t+1)\exp(-\lambda e_{\max} t)}\right)\alpha_3 \end{array}\right] \\ & - \frac{c}{\lambda}(1 - \exp(-\lambda e_{\max}(T-t))) - \delta(T-t) \end{aligned}$$

is the payoff in the event that the announcement is not met with an implementation decision by the other player. In this case, the updated beliefs about the value of the project are

$$\frac{\exp(-\lambda e_{\max} T)}{(\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t)}\alpha_2 + \left(1 - \frac{\exp(-\lambda e_{\max} T)}{(\lambda e_{\max} t + 1)\exp(-\lambda e_{\max} t)}\right)\alpha_3,$$

where $\frac{\exp(-\lambda e_{\max} T)}{(\lambda e_{\max} t+1)\exp(-\lambda e_{\max} t)}$ and $1 - \frac{\exp(-\lambda e_{\max} T)}{(\lambda e_{\max} t+1)\exp(-\lambda e_{\max} t)}$ are the updated beliefs that the other player will have produced 0 and 1 breakthrough by the deadline given that they have not implemented the project upon receiving the announcement and hence do not have 2 breakthroughs.

Proposition 10. $\exists 0 < T < \tilde{X} : V_i(t,1,0,0) \geq \zeta$.

Proof. Rearranging $V_i(t, 1, 0, 0) \geq \zeta$ using the expressions above, we obtain

$$\frac{\frac{\lambda e_{\max} t}{\lambda e_{\max} t+1}\left(1-\exp\left(-\lambda e_{\max}(T-t)\right)\right)+\frac{1}{\lambda e_{\max} t+1}\left(1-\left(\lambda e_{\max}(T-t)+1\right)\exp(-\lambda e_{\max}(T-t))\right)}{1-\left(\lambda e_{\max} t+1\right)\exp(-\lambda e_{\max} t)}$$
$$\geq \frac{\delta\,(T-t)+\left(\frac{c}{\lambda}-\alpha_4\right)\left(1-\exp(-\lambda e_{\max}(T-t))\right)}{\left(\lambda e_{\max} t+1\right)\exp(-\lambda e_{\max} t)\times\left[\exp\left(-\lambda e_{\max}(T-t)\right)\alpha_3+\left(1-\exp(-\lambda e_{\max}(T-t))\right)\alpha_4\right]}$$

This holds with equality for $t = T$ and holds strictly for $t = 0$. It suffices to check that for any $t \in (0, T)$ the following holds,

$$\frac{\lambda e_{\max} t \exp(-\lambda e_{\max} t)}{1-\left(\lambda e_{\max} t+1\right)\exp(-\lambda e_{\max} t)} \geq \frac{\delta}{\alpha_3}\frac{T-t}{1-\exp\left(-\lambda e_{\max}(T-t)\right)}+\frac{\frac{c}{\lambda}-\alpha_4}{\alpha_3}.$$

Since $\frac{(1-\exp(-\lambda e_{\max}(T-t)))}{T-t} < 1$, this condition is implied by

$$\frac{\exp(-\lambda e_{\max} t)-\exp\left(-\lambda e_{\max} T\right)}{T-t}\frac{\lambda e_{\max} t}{1-\left(\lambda e_{\max} t+1\right)\exp(-\lambda e_{\max} t)} \geq \frac{\delta}{\alpha_3}+\frac{\frac{c}{\lambda}-\alpha_4}{\alpha_3}. \tag{7}$$

Note that the RHS of (7) is clearly finite. Now consider the first term on the LHS of (7),

$$\frac{\exp(-\lambda e_{\max} t)-\exp\left(-\lambda e_{\max} T\right)}{T-t}.$$

This term is decreasing in t and thus has a finite positive lower bound of $\frac{1-\exp(-\lambda e_{\max} T)}{T}$ which in the limit $T \to 0$ approaches $\lambda e_{\max}$. Now consider the limit of the second term on the LHS of (7),

$$\lim_{t\to 0}\frac{\lambda e_{\max} t}{\left(1-\left(\lambda e_{\max} t+1\right)\exp(-\lambda e_{\max} t)\right)}=\infty$$

Hence $\exists T > 0$ such that (7) is satisfied for all $0 < t \leq T$. ☐

B.4.3 Infinite Deadline

To gain some intuition for what may occur a long way from the deadline we consider an equilibrium of the infinite horizon game. This is done to avoid the complications in a game with a finite horizon, of specifying the changes in behaviour for the subgames as we transition from behavior far away from the deadline to close to the deadline. The infinite horizon case allows one to focus on a setting where there is no effect of a future deadline. We show that the equilibrium strategies are similar to the equilibrium strategies in our earlier model far away from the deadline ($t < T - Y$). We find there is immediate revelation of private information about breakthroughs by each individual, $w_i^* = n_i$. Hence, the project is implemented whenever the combined number of breakthroughs reaches 2, $d_i^* = implement$ if $n_i + w_{-i} \geq 2$. This is similar to the earlier model whereby the agents immediately implement the project upon a breakthrough. Individuals also exert less than the maximum effort level $e_i^* = \frac{\delta}{c} < e_{\max}$ as in the earlier model which trades off freeriding incentives with incentives to bring forward the implementation of the project. We formalize this tradeoff in the following proposition.

Proposition 11. *A symmetric perfect Bayesian equilibrium of the infinite horizon game is*

$$\begin{aligned}
w_i^*(t, n_i) &= 1 \text{ if } n_i \geq 1 \\
d_i^*(t, n_i, \mu, \nu) &= \begin{cases} \text{implement if } \begin{cases} n_i \geq 1 \text{ and } \mu = -i \\ n_i = 2 \end{cases} \\ 0 \text{ otherwise} \end{cases} \\
e_i^*(t, n_i, \mu, \nu) &= \begin{cases} 0 \text{ if } \begin{cases} n_i \geq 1 \text{ and } \mu = -i \\ n_i = 2 \end{cases} \\ \frac{\delta}{c} \text{ otherwise} \end{cases} \\
\phi_i^*(t, \mu, \nu) &= \begin{cases} \{0, 1, 0\} \text{ if } \mu = -i \\ 0 \text{ otherwise} \end{cases}
\end{aligned}$$

Proof. The continuation values are

$$\begin{aligned}
V_i(t, 2, i, \nu) &= V_0 + \alpha_1 + \alpha_2 \\
V_i(t, 2, -i, \nu) &= V_0 + \alpha_1 + \alpha_2 + \alpha_3 \\
V_i(t, 2, 0, 0) &= V_0 + \alpha_1 + \alpha_2 \\
V_i(t, 1, i, \nu) &= V_0 + \alpha_1 + \alpha_2 - \frac{c}{\lambda} \\
V_i(t, 1, -i, \nu) &= V_0 + \alpha_1 + \alpha_2 \\
V_i(t, 1, 0, 0) &= V_0 + \alpha_1 + \alpha_2 - \frac{c}{\lambda} \\
V_i(t, 0, -i, \nu) &= V_0 + \alpha_1 + \alpha_2 - \frac{c}{\lambda} \\
V_i(t, 0, 0, 0) &= V_0 + \alpha_1 + \alpha_2 - \frac{2c}{\lambda}.
\end{aligned}$$

As soon as an agent knows that a combined two breakthrough have been produced this results in zero effort as $\alpha_3 < \frac{c}{\lambda}$. This is true when $n_i = 2$ or when $n_i = 1$ and $\mu = -i$. Otherwise, this non-zero effort strategy is optimal provided that

$$\begin{aligned}
V_i(t, 2, i, \nu) - V_i(t, 1, i, \nu) &= \frac{c}{\lambda} \\
V_i(t, 2, 0, 0) - V_i(t, 1, 0, 0) &= \frac{c}{\lambda} \\
V_i(t, 1, 0, 0) - V_i(t, 0, 0, 0) &= \frac{c}{\lambda} \\
V_i(t, 1, -i, \nu) - V_i(t, 0, -i, \nu) &= \frac{c}{\lambda},
\end{aligned}$$

which is straightforward to verify from inspection of the continuation values. The decision not to implement the project is optimal provided that

$$\begin{aligned}
V_i(t, 1, 0, 0) &\geq V_0 + \alpha_1 \\
V_i(t, 1, i, \nu) &\geq V_0 + \alpha_1 \\
V_i(t, 0, -i, \nu) &\geq V_0 + \alpha_1 \\
V_i(t, 0, 0, 0) &\geq V_0
\end{aligned}$$

which is straightforward to verify. For the history $(t, 1, -i, \nu)$, the decision to implement is optimal provided that for all $s \geq t$

$$\begin{aligned} V_i(t,1,-i,\nu) &\geq \int_t^s (V_0+\alpha_1+\alpha_2+\alpha_3-\delta(r-t))\lambda\frac{\delta}{c}\exp\left(-\lambda\frac{\delta}{c}(r-t)\right)dr \\ &\quad +\exp\left(-\lambda\frac{\delta}{c}(s-t)\right)[V_i(s,1,-i,\nu)-\delta(s-t)] \\ &= \left[V_0+\alpha_1+\alpha_2+\alpha_3-\frac{c}{\lambda}\right]\left(1-\exp\left(-\lambda\frac{\delta}{c}(s-t)\right)\right)+(V_0+\alpha_1+\alpha_2)\exp\left(-\lambda\frac{\delta}{c}(s-t)\right), \end{aligned}$$

which follows by $\frac{c}{\lambda} \geq \alpha_3$. An almost identical condition holds for $(t,2,i,\nu)$ and $(t,2,-i,\nu)$. A very similar condition is also derived for the history $(t,2,0,0)$:

$$\begin{aligned} V_i(t,2,0,0) &\geq \int_t^s (V_i(r,2,-i,\nu)-\delta(r-t))\lambda\frac{\delta}{c}\exp\left(-\lambda\frac{\delta}{c}(r-t)\right)dr \\ &\quad +\exp\left(-\lambda\frac{\delta}{c}(s-t)\right)[V_i(s,2,0,0)-\delta(s-t)] \\ &= \left[V_0+\alpha_1+\alpha_2+\alpha_3-\frac{c}{\lambda}\right]\left(1-\exp\left(-\lambda\frac{\delta}{c}(s-t)\right)\right)+(V_0+\alpha_1+\alpha_2)\exp\left(-\lambda\frac{\delta}{c}(s-t)\right), \end{aligned}$$

which also follows by $\frac{c}{\lambda} \geq \alpha_3$. The announcement strategy is optimal provided that

$$\begin{aligned} V_i(t,1,i,t) &\geq V_i(t,1,0,0) \\ V_i(t,2,i,t) &\geq V_i(t,2,0,0), \end{aligned}$$

both of which hold with equality. $\square$