Online Appendix to Accompany "Marginal Jobs, Heterogeneous Firms, and Unemployment Flows"

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A Solution of the Simulated Model

Here we present technical details of the solution to the model in sections 2 and 3 for the purposes of the quantitative applications in section 4. For simplicity, we present the solution approach for a given fixed firm productivity φ, so we supress this notation in what follows. Aggregation across φ is achieved simply by integrating over the known distribution of φ.

Steady State Optimal Employment Policy Idiosyncratic shocks evolve according to (17) with $x \sim Pareto\,(1-k_x^{-1}, k_x)$. Denoting the distribution function of x as $\tilde{G}$, we can rewrite the recursion for the function $D\,(n,x)$ in Proposition 3 as:

$$\begin{aligned} D\,(n,x) \;&=\; (1-\lambda)\,\chi\,(x) + \lambda \int_{R(n)}^{R_v(n)} \chi\,(x')\, d\tilde{G}\,(x') + \lambda \int_{R_v(n)}^{\infty} \frac{c}{q} d\tilde{G}\,(x') \\ &\quad + \beta\,(1-\lambda)\, D\,(n,x) + \beta\lambda \int_{R(n)}^{R_v(n)} D\,(n,x')\, d\tilde{G}\,(x')\,, \end{aligned} \tag{1}$$

where $\chi\,(x) \equiv (1-\eta)\left[\frac{px\alpha n^{\alpha-1}}{1-\eta(1-\alpha)} - b\right] - \eta\beta f \frac{c}{q}$. We conjecture that the function $D\,(n,x)$ is of the form $D\,(n,x) = d_0 + d_1\chi\,(x)$. Substituting this into the latter, and equating coefficients, we obtain the following solution for $D(n,x)$:

$$\begin{aligned} D(n,x) &= \frac{1-\lambda}{1-\beta(1-\lambda)}\chi(x) \\ &+\frac{\lambda}{1-\beta(1-\lambda)}\frac{\tilde{G}[R_v(n)]-\tilde{G}[R(n)]}{1-\beta(1-\lambda)-\beta\lambda\left(\tilde{G}[R_v(n)]-\tilde{G}[R(n)]\right)}\mathcal{Q}(n) \\ &+\frac{1-\tilde{G}[R_v(n)]}{1-\beta(1-\lambda)-\beta\lambda\left(\tilde{G}[R_v(n)]-\tilde{G}[R(n)]\right)}\lambda\frac{c}{q}, \end{aligned} \tag{2}$$

where $\mathcal{Q}(n) \equiv \mathbb{E}(\chi(x')\,|x' \in [R(n), R_v(n)])$. Substituting into the first order conditions for hires and separations (13) and (14) yields two nonlinear equations in the optimal employment policy $R(n)$ and $R_v(n)$ that are straightforward to solve numerically. The aggregate employment stock and flows are then obtained directly from applying the results of Proposition 5.

Average Product and Average Marginal Product The average product of labor implied by the model is given by $APL = \mathbb{E}[pxn^{\alpha-1}]$. Note that:

$$\mathbb{E}\left[xn^{\alpha-1}\right] = \int\left[\int xdG(x|n)\right]n^{\alpha-1}dH(n).$$

Moreover, the optimal employment policy implies that, given n, x must lie in the interval $[R(n), R_v(n)]$, but is otherwise independently distributed. Thus:

$$\int xdG(x|n) = \frac{\int_{R(n)}^{R_v(n)} xdG(x)}{G[R_v(n)]-G[R(n)]}. \tag{3}$$

Thus:

$$APL = \mathbb{E}\left[pxn^{\alpha-1}\right] = \int\frac{\int_{R(n)}^{R_v(n)} xdG(x)\,n^{\alpha-1}}{G[R_v(n)]-G[R(n)]}dH(n). \tag{4}$$

The average marginal product of labor is simply given by $\mathbb{E}[MPL] = \mathbb{E}[px\alpha n^{\alpha-1}] = \alpha APL$.

Average Wages It follows from equation (9) that the average wage across firms is given by:

$$\bar{w}_f = \frac{\eta}{1-\eta(1-\alpha)}\mathbb{E}[MPL] + \eta\beta f\frac{c}{q} + (1-\eta)b. \tag{5}$$

To obtain the average wage across workers, which we denote $\bar{w}_w$, note that $\bar{w}_w = \mathbb{E}\left[\frac{n}{\mathbb{E}(n)} w(n,x)\right]$ where $w(n,x)$ is the wage in a given firm defined in (9). That is, it is the employment-weighted average of wages across firms. Thus:

$$\bar{w}_w = \frac{\eta}{1-\eta(1-\alpha)} \frac{1}{\mathbb{E}(n)} \mathbb{E}\left[px\alpha n^{\alpha}\right] + \eta\beta f \frac{c}{q} + (1-\eta) b. \tag{6}$$

This has a very similar structure to the average wage across firms. It follows that:

$$\bar{w}_w = \frac{\eta p \alpha}{1-\eta(1-\alpha)} \frac{1}{\mathbb{E}(n)} \int \frac{\int_{R(n)}^{R_v(n)} x dG(x) n^{\alpha}}{G[R_v(n)] - G[R(n)]} dH(n) + \eta\beta f \frac{c}{q} + (1-\eta) b. \tag{7}$$

Finally, the average wage of new hires, which we denote $\bar{w}_m$, is equal to a hiring-weighted average of wages across hiring firms. Noting from (12) that idiosyncratic productivity of hiring firms is given by $x = R_v(n)$, we have that:

$$\bar{w}_m = \mathbb{E}\left[\mathbb{E}\left(w(n,x) | n > n_{-1}, n_{-1}\right)\right] = \int \int_{n_{-1}} w(n, R_v(n)) \frac{dG[R_v(n)]}{1 - G[R_v(n_{-1})]} dH(n_{-1}). \tag{8}$$

B Stationary AR(1) Aggregate Shocks

In the main text, we take a first-order approximation to the firm's first-order conditions around the stochastic steady state. We show that, under the assumption that aggregate productivity follows a symmetric random walk, one obtains an analytical solution for the optimal policy rule out of steady state.

We now investigate the robustness of our approximate solution. Rather than linearize, we numerically solve the nonlinear optimal policy problem. In addition, since the random-walk assumption was made to facilitate solution of the linear approximation, we also take this opportunity to consider the case of stationary aggregate productivity. Specifically, we assume that productivity, p, obeys an AR(1),

$$p' = (1-\gamma) + \gamma p + \varepsilon', \text{ where } \varepsilon' \sim N\left(0, \sigma^2\right). \tag{9}$$

The details of our approach are as follows. As noted in the main text, firms must forecast market tightness in order to forecast their future wages. Market tightness depends, in turn, on the aggregation of labor demands across firms, that is, on the entire distribution of employment across employers. As in the main text, we therefore adopt an approach akin to that in Krusell and Smith (1998) and conjecture that, with respect to market tightness, the first moment of the distribution is the only quantitatively important one. We assume firms condition their decisions on the linear rule,

$$\theta = \theta_0 + \theta_N N + \theta_p p. \tag{10}$$

This may be regarded as a special case of the forecast rule (26) used in the main text. Since aggregate productivity is stationary in this case, the forecast rules are better thought of as linear approximations around the *non*-stochastic steady state (with $p = 1$). Thus, in contrast to the forecast rules in (26), steady-state employment and labor market tightness (N^* and θ^*) do not vary and so are absorbed into a constant.

It follows that, to forecast market tightness, one must forecast aggregate productivity and mean employment, N. Equation (10) gives the law of motion of aggregate productivity. Again as in the main text (and in the spirit of Krusell and Smith), we assume firms use a linear autoregressive forecast rule for mean employment,

$$N' = \nu_0 + \nu_N N + \nu_p p'. \tag{11}$$

Given (9) through (11), the firm's dynamic program is given by

$$\begin{aligned} &\Pi\left(n_{-1}, x, p, N, \theta\right) \\ &= \max_n \Bigg\{ pxn^\alpha - w\left(n, x, p\right) n - \frac{c}{q\left(\theta\right)} \Delta n \mathbf{1}^+ \\ &\quad + \beta \int \int \Pi\left(n, x', p', N'\left(p', N\right), \theta'\left(p', N\right)\right) g\left(x'|x\right) \phi\left(p'|p\right) dx' dp' \Bigg\}. \end{aligned} \tag{12}$$

The problem is conditioned on the current aggregate state, which is summarized by the triple, (p, N, θ). Note, though, that, for any given (n, x'), the forward value may be predicted from (p', N); the future value of market tightness and employment are summarized entirely by this pair. Given a calibration of (9) and an initial guess for the coefficients in (10) and (11), value iteration on Π yields the optimal (inverse) labor demand rules, R and R_v.

Importantly, aggregation of the policy rules proceeds just as before. This is a significant point to note: the law of motion (31) is valid *for any* given pair, (R, R_v). Thus, we do not have to simulate individual firms. This speeds up the calculation of aggregate equilibrium and reduces the amount of noise in the aggregates; the latter is not necessarily a small concern when idiosyncratic risk is large, as it is in our calibration.[1]

[1] As Algan, Allais, and den Haan (2008) note, sampling variation vanishes at a rate proportional to the square root of the sample size, so very large samples are typically needed in Krusell and Smith applications.

Thus, we use (31) to simulate the model economy.[2] We then estimate the linear rules (10) and (11) on the simulated data in order to check the accuracy of the initial guess on $\{\theta_0, \theta_N, \theta_p\}$ and $\{\nu_0, \nu_N, \nu_p\}$. We use the regression results to update these parameters and repeat until convergence. The algorithm finds that

$$\begin{aligned} \hat{\theta}_0 = -6.38, \quad \hat{\theta}_N = 1.338, \quad \hat{\theta}_p = 2.664, \\ \hat{\nu}_0 = 0.176, \quad \hat{\nu}_N = 0.93, \quad \hat{\nu}_p = 0.054. \end{aligned} \tag{13}$$

The R^2 associated with (11) is 0.9999. The R^2 associated with (10) is somewhat lower, at 0.998.[3]

Our calibration of (9) sets $\gamma = 0.9925$ and $\sigma = 0.00336$. This calibration of the weekly aggregate productivity process allows the model to nearly replicate the quarterly time series properties of output per worker. To be specific, we take quarterly averages of the weekly simulated data on employment and output, divide the latter by the former, HP detrend, and calculate the autocorrelation and standard deviation. The former is 0.917, and the latter is 0.018. These are quite similar to the empirical autocorrelation (0.88) and standard deviation (0.02) of output per worker.

We use the simulated data to compute the elasticities of labor market variables with respect to output per worker. For instance, we regress the model-generated (HP-filtered) time series of the job finding rate on model-generated (HP-filtered) output per worker. This reveals an elasticity of 2.66, which is almost identical to that observed in BLS data. However, the elasticity of the inflow rate is -1.29, which is somewhat smaller than observed and also lower than what we find in the model with random-walk aggregate productivity.

We interpret these results as follows. In regards to inflows, firms react more aggressively to aggregate shocks on the separation margin when firms know the shock will persist. This accounts for the reduced cyclicality of the inflow rate. It would seem that this intuition also applies to hires: if shocks are temporary, they should have less effect on the present discounted value of a worker to a firm. Thus, hires should not change by as much as in the case with non-stationary aggregate productivity.

[2] The horizon for the simulation is 75 years (exclusive of a five-year burn-in period). We found that the results were similar if we increased the length of the simulation. We should note that this simulation was performed on the model without fixed effects in order to accelerate computation time. (One would have to compute the policy rules for each fixed effect.) We have verified that, in the baseline model (in which p follows a random walk), the cyclical properties are essentially invariant to the inclusion of the fixed effects.

[3] We acknowledge that R^2 has shortcomings as a measure of forecast accuracy. For instance, a high R^2 indicates an accurate one-step-ahead forecast, but the reliability of the forecast rule may decline at longer horizons. Notwithstanding recent research into this (see, for example, den Haan, 2010), the literature on heterogeneous-agent general equilibrium models has not settled on a clear alternative.

This intuition with respect to hires is valid in partial equilibrium, but there is an additional, general equilibrium consideration with regard to vacancy creation in particular. To see this, suppose there is a negative aggregate shock. Consider a firm's decision problem immediately after the "burst" of inflows but before vacancies are posted. Since there are fewer separations (in light of the stationarity of p), the immediate rise in unemployment is somewhat lessened, and so the cost of a hire $c/q(\theta)$ falls less than in the case with random-walk productivity. This of course reduces the firm's incentive to create vacancies. Thus, although stationarity implies that the incentive to hire changes by less *conditional on* θ, the adjustment in the price of a new hire will tend to put downward pressure on hires (relative to the calibration with greater persistence in p). As a result, this firm (and other firms in its position) will post fewer vacancies than otherwise. It is possible that the labor market settles into an equilibrium such that θ actually falls by as much, or more, than in the case with greater persistence.

Lastly, in Figure A.1, we display the impulse responses of the labor market flows to a one percent decline in aggregate productivity, p. The figure is derived by simulating the model at a weekly frequency, computing the monthly average of the variables, and expressing each variable relative to its steady state. The job-finding rate and market tightness each display a hump-shaped response, with the decline reaching a peak between one and two quarters after the shock. This degree of persistence is a significant improvement over Mortensen and Pissarides (1994), in which f is a jump variable. The model does understate, though, the persistence observed in the data. Michelacci and Lopez-Salido (2007) find that f does not begin its return to steady state until sometime in the third quarter following the shock. Fujita and Ramey (2006) find an even more persistent response in regards to market tightness.

C Omitted Proofs

Proof of Proposition 2. First we verify that, under Assumption 1, Π is concave and supermodular. The argument applies standard recursive methods. The operator on the right side of the Bellman equation (3) is a contraction. Hence, there is a unique fixed point within the space, $\mathcal{C}$, of bounded and continuous functions which solves (3).[4] One may show, moreover, that the operator maps the space of concave and supermodular functions into itself, so the fixed point must lie within this subset of $\mathcal{C}$.[5] Since the fixed point is unique, it

[4] Flow profit is not naturally bounded. But one may impose bounds on the stochastic process of x and then suitably reinterpret G as the c.d.f of the truncated process.

[5] The analyses in Dixit (1997) and Eberly and van Mieghem (1997) illustrate this argument in a model of kinked adjustment costs. Please see their papers for details.

follows that Π is concave and supermodular.

Concavity of Π guarantees that the first-order conditions (4) are sufficient to identify the global maximum, conditional on adjusting. It is helpful to state the first-order conditions in terms of the function

$$\tilde{\Pi}(n,x) = \pi(n,x) + \beta \int \Pi(n,x')\, dG(x'|x), \tag{14}$$

where $\pi(n,x) \equiv pxF(n) - w(n,x)n$ is flow profit (gross of the adjustment cost), which is assumed to be strictly concave. Note that since the sum of weakly (Π) and strictly (π) concave functions is strictly concave, $\tilde{\Pi}$ is strictly concave. The first-order conditions are then

$$\begin{aligned} \Delta n &> 0 : \tilde{\Pi}_n(n,x) = \frac{c}{q(\theta)}, \\ \Delta n &< 0 : \tilde{\Pi}_n(n,x) = 0. \end{aligned} \tag{15}$$

The latter yield unique values of the (inverse) labor demand functions, $R(n)$ and $R_v(n)$. Note that $\tilde{\Pi}_n$ is increasing in x by supermodularity and that, for any values of employment $n_2 > n_1$, $\tilde{\Pi}_n(n_2,x)$ lies everywhere below $\tilde{\Pi}_n(n_1,x)$ by concavity. It follows that R and R_v are unique; that $R_v > R$ for any n; and that R and R_v are each increasing in n.

Lastly, with respect to the decision to adjust, it is apparent that if the marginal value of labor, evaluated at n_{-1}, exceeds the marginal cost of adjusting, the firm ought to adjust. Thus, the firm hires if and only if $\tilde{\Pi}_n(n_{-1}, R_v(n_{-1})) > c/q(\theta)$. By supermodularity, this implies $x > R_v(n_{-1})$. The analogous rule holds for the decision to separate. ■

Proof of Proposition 3. First, note that one can re-write the continuation value conditional on each of the three possible continuation regimes:

$$\Pi(n,x') = \begin{cases} \Pi^{-}(n,x') & \text{if} \quad x' < R(n), \\ \Pi^{0}(n,x') & \text{if} \quad x' \in [R(n), R_v(n)], \\ \Pi^{+}(n,x') & \text{if} \quad x' > R_v(n), \end{cases} \tag{16}$$

where superscripts $^{-/0/+}$ refer to whether their are separations, a hiring freeze, or hires tomorrow. Thus we can write[6]:

$$\int \Pi(n,x')\,dG(x'|x) = \int_0^{R(n)} \Pi^-(n,x')\,dG + \int_{R(n)}^{R_v(n)} \Pi^0(n,x')\,dG + \int_{R_v(n)}^{\infty} \Pi^+(n,x')\,dG. \tag{17}$$

Taking derivatives with respect to n, recalling the definition of $D(\cdot)$, and noting that, since $\Pi(n,x')$ is continuous, it must be that $\Pi^-(n,R(n)) = \Pi^0(n,R(n))$ and $\Pi^0(n,R_v(n)) = \Pi^+(n,R_v(n))$, yields:

$$D(n,x) = \int_0^{R(n)} \Pi_n^-(n,x')\,dG + \int_{R(n)}^{R_v(n)} \Pi_n^0(n,x')\,dG + \int_{R_v(n)}^{\infty} \Pi_n^+(n,x')\,dG. \tag{18}$$

Finally, using the Envelope conditions in Lemma 1 below, and substituting into (18) we obtain (15) and (16) in the main text:

$$\begin{aligned} D(n,x) &= \int_{R(n)}^{R_v(n)} \left\{(1-\eta)\left[\frac{px'\alpha n^{\alpha-1}}{1-\eta(1-\alpha)} - b\right] - \eta\beta f\frac{c}{q}\right\} dG(x'|x) \\ &\quad + \int_{R_v(n)}^{\infty} \frac{c}{q} dG(x'|x) + \beta \int_{R(n)}^{R_v(n)} D(n,x')\,dG(x'|x) \\ &\equiv (\mathbf{C}D)(n,x). \end{aligned} \tag{19}$$

To verify that $\mathbf{C}$ is a contraction mapping, we confirm that Blackwell's sufficient conditions for a contraction hold here (see Stokey and Lucas 1989, p.54). To verify monotonicity, fix $(n,x) = (\bar{n},\bar{x})$, and take $\hat{D} \geq D$. Then note that:

$$\begin{aligned} &\int_{R(\bar{n})}^{R_v(\bar{n})} \hat{D}(\bar{n},x')\,dG(x'|\bar{x}) - \int_{R(\bar{n})}^{R_v(\bar{n})} D(\bar{n},x')\,dG(x'|\bar{x}) \\ &\qquad = \int_{R(\bar{n})}^{R_v(\bar{n})} \left[\hat{D}(\bar{n},x') - D(\bar{n},x')\right] dG(x'|\bar{x}) \geq 0. \end{aligned} \tag{20}$$

Since $(\bar{n},\bar{x})$ were arbitrary, it thus follows that $\mathbf{C}$ is monotonic in D. To verify discounting, note that:

$$[\mathbf{C}(D+a)](n,x) = (\mathbf{C}D)(n,x) + \beta a\,[G(R_v(n)|x) - G(R(n)|x)] \leq (\mathbf{C}D)(n,x) + \beta a. \tag{21}$$

Since $\beta < 1$ it follows that $\mathbf{C}$ is a contraction. It therefore follows from the Contraction Mapping Theorem that $\mathbf{C}$ has a unique fixed point. ■

[6] Henceforth, "dG" without further elaboration is to be taken as "$dG(x'|x)$".

Lemma 1 *The value function defined in (3) has the following properties:*

$$\begin{aligned}
\Pi_n^-(n,x') &= 0, \\
\Pi_n^0(n,x') &= (1-\eta)\left[\frac{px'\alpha n^{\alpha-1}}{1-\eta(1-\alpha)}-b\right]-\eta\beta f\frac{c}{q}+\beta D(n,x'), \\
\Pi_n^+(n,x') &= c/q.
\end{aligned} \tag{22}$$

Proof of Lemma 1. First, note that standard application of the Envelope Theorem implies that $\Pi_n^-(n,x')=0$ and $\Pi_n^+(n,x')=c/q$. It is only slightly less obvious what happens when $\Delta n'=0$, i.e. when the employment is frozen next period. In this case, $n'=n$ and this implies that:

$$\Pi^0(n,x') = px'F(n) - w(n,x')n + \beta\int \Pi(n,x'')\,dG(x''|x'). \tag{23}$$

It therefore follows that:

$$\Pi_n^0(n,x') = px'F'(n) - w(n,x') - w_n(n,x')n + \beta\int \Pi_n(n,x'')\,dG(x''|x'). \tag{24}$$

Since, by definition $D(n,x') \equiv \int \Pi_n(n,x'')\,dG(x''|x')$, the statement holds as required. ■

Proof of Proposition 4. First note that if x evolves according to (17), then we can rewrite the recursion for $D(n,x)$ as:

$$\begin{aligned}
D(n,x) = {} & \frac{1-\lambda}{1-\beta(1-\lambda)}\chi(x) + \frac{\lambda}{1-\beta(1-\lambda)}\int_{R(n)}^{R_v(n)}\chi(x')\,d\tilde{G}(x') \\
& + \frac{\lambda}{1-\beta(1-\lambda)}\int_{R_v(n)}^{\infty}\frac{c}{q}d\tilde{G}(x') + \frac{\beta\lambda}{1-\beta(1-\lambda)}\int_{R(n)}^{R_v(n)}D(n,x')\,d\tilde{G}(x'),
\end{aligned} \tag{25}$$

where $\chi(x) \equiv (1-\eta)\left[\frac{px\alpha n^{\alpha-1}}{1-\eta(1-\alpha)}-b\right]-\eta\beta c\theta$. It follows that the LHS of the first-order conditions, (13) and (14) are increasing in x, because $\chi(x)$ is increasing in x. Thus, to establish that $\partial R_v/\partial p<0$ and $\partial R/\partial p<0$, simply note that the function $D(n,x)$ is also increasing in p and thus the LHS of (13) and (14) are increasing in p.

To ascertain the marginal effects of θ we first need to establish the marginal effect of θ on the function $D(n,x)$. Rewriting $f/q=\theta$ and $q=q(\theta)$ in (25), differentiating with respect to θ, and using the first-order conditions, (13) and (14), to eliminate terms we obtain:

$$D_\theta = -\eta\beta c\frac{1-\lambda(1-p^0)}{1-\beta[1-\lambda(1-p^0)]} - \frac{c}{q}\frac{q'(\theta)}{q}\frac{\lambda p^+}{1-\beta[1-\lambda(1-p^0)]}, \tag{26}$$

where $p^0 \equiv \tilde{G}(R_v(n)) - \tilde{G}(R(n))$, $p^+ \equiv 1 - \tilde{G}(R_v(n))$, and $p^- \equiv \tilde{G}[R(n)]$. Note that D_θ is independent of x. Differentiating the first-order condition for a hiring firm, (13), with respect to θ we obtain:

$$-\eta\beta c + \frac{c}{q}\frac{q'(\theta)}{q} + \beta D_\theta = -\frac{\eta\beta c}{1-\beta[1-\lambda(1-p^0)]} + \frac{c}{q}\frac{q'(\theta)}{q}\frac{1-\beta(1-\lambda p^-)}{1-\beta[1-\lambda(1-p^0)]} < 0 \quad (27)$$

since $q'(\theta) < 0$. Thus it follows that $\partial R_v/\partial\theta > 0$. Likewise, differentiating the first-order condition for a shedding firm, (14), with respect to θ we obtain:

$$-\eta\beta c + \beta D_\theta = -\frac{\eta\beta c}{1-\beta[1-\lambda(1-p^0)]} - \beta\frac{c}{q}\frac{q'(\theta)}{q}\frac{\lambda p^+}{1-\beta[1-\lambda(1-p^0)]}. \quad (28)$$

Thus, $\partial R/\partial\theta > 0 \iff n > R_v^{-1}\tilde{G}^{-1}\left(1 + \frac{\eta}{\varepsilon_{q\theta}}\frac{f}{\lambda}\right)$ where $\varepsilon_{q\theta} \equiv \frac{d\ln q}{d\ln\theta}$. ■

Proof of Proposition 7. Consider the c.d.f. of employment growth for a given lagged employment level, n_{-1}, and for the case where employment growth is negative:

$$\begin{aligned}\Pr(\Delta\ln n < \delta | n_{-1}, \delta < 0) &= \Pr\left(\ln R^{-1}(x) - \ln n_{-1} < \delta | n_{-1}\right) \\ &= \Pr\left(x < R\left(e^\delta n_{-1}\right) | n_{-1}\right) \\ &= \lambda\tilde{G}\left[R\left(e^\delta n_{-1}\right)\right].\end{aligned} \quad (29)$$

It follows that the unconditional c.d.f. of employment growth, given that $\Delta\ln n < 0$ is equal to:

$$H_\Delta(\delta) \equiv \Pr(\Delta\ln n < \delta) = \lambda\int\tilde{G}\left[R\left(e^\delta n_{-1}\right)\right]dH(n_{-1}), \quad (30)$$

It follows that the density of employment growth is given by

$$h_\Delta(\delta) = H'_\Delta(\delta) = \lambda\int\tilde{G}'\left[R'\left(e^\delta n_{-1}\right)\right]e^\delta n_{-1}dH(n_{-1}), \quad (31)$$

as stated in the Proposition. A similar method reveals that, in the case where $\Delta\ln n > 0$:

$$H_\Delta(\delta) = \lambda\int\tilde{G}\left[R_v\left(e^\delta n_{-1}\right)\right]dH(n_{-1}), \text{ and } h_\Delta(\delta) = \lambda\int\tilde{G}'\left[R'_v\left(e^\delta n_{-1}\right)\right]e^\delta n_{-1}dH(n_{-1}). \quad (32)$$

Finally there is a mass point at zero employment growth. Clearly that is given by:

$$h_\Delta(0) = H_\Delta\left(0^+\right) - H_\Delta\left(0^-\right) = \lambda\int\left(\tilde{G}[R_v(n_{-1})] - \tilde{G}[R(n_{-1})]\right)dH(n_{-1}). \quad (33)$$

■

Lemma 2 *If idiosyncratic shocks evolve according to (17), and the matching function is of the form* $M(U,V) = \mu U^{\phi} V^{1-\phi}$*, then the marginal firm surplus defined in (36) is given by*

$$
\begin{aligned}
J &= \frac{\psi p \alpha n^{\alpha-1}}{1-\beta(1-\lambda)}\left[x + \frac{\beta\lambda p^0}{1-\beta(1-\lambda)-\beta\lambda p^0}\mathcal{E}(n)\right] \\
&\quad - \frac{(1-\eta)b}{1-\beta(1-\lambda)-\beta\lambda p^0} - \beta\frac{c}{q}\frac{\eta f - \lambda p^+}{1-\beta(1-\lambda)-\beta\lambda p^0}, \qquad (34)
\end{aligned}
$$

and the marginal effects of n*,* p *and* θ *on* J *are given by*

$$
\begin{aligned}
J_n &= -\frac{1-\alpha}{n}\frac{\psi p \alpha n^{\alpha-1}}{1-\beta(1-\lambda)}\left[x + \frac{\beta\lambda p^0}{1-\beta(1-\lambda)-\beta\lambda p^0}\mathcal{E}(n)\right] \\
J_p &= \frac{1}{p}\frac{\psi p \alpha n^{\alpha-1}}{1-\beta(1-\lambda)}\left[x + \frac{\beta\lambda p^0}{1-\beta(1-\lambda)-\beta\lambda p^0}\mathcal{E}(n)\right] \\
J_\theta &= -\beta\frac{c}{q}\frac{1}{\theta}\frac{\eta f - \phi\lambda p^+}{1-\beta(1-\lambda)-\beta\lambda p^0}, \qquad (35)
\end{aligned}
$$

where $\psi \equiv \frac{1-\eta}{1-\eta(1-\alpha)}$*,* $\mathcal{E}(n) \equiv \mathbb{E}(x'|x' \in [R(n), R_v(n)])$*, and* p^0*,* p^+ *are as defined in the Proof to Proposition 4.*

Proof. Since firms only receive an idiosyncratic shock with probability λ each period, we can use the recursion for $J(n,x)$, (36), to write:

$$
\begin{aligned}
J(n,x) &= \frac{1}{1-\beta(1-\lambda)}\left[\psi p x \alpha n^{\alpha-1} - (1-\eta)b - \eta\beta c\theta\right] \\
&\quad + \frac{\beta\lambda}{1-\beta(1-\lambda)}\frac{c}{q}\int_{R_v(n)} d\tilde{G} + \frac{\beta\lambda}{1-\beta(1-\lambda)}\int_{R(n)}^{R_v(n)} J(n,x')\,d\tilde{G}. \qquad (36)
\end{aligned}
$$

We then conjecture that $J(n,x)$ is of the form $j_0 + j_1 x$. Substituting this assumption into the latter, and equating coefficients yields:

$$
\begin{aligned}
j_0 &= -\frac{(1-\eta)b}{1-\beta(1-\lambda)} - \beta\frac{c}{q}\frac{\eta f - \lambda p^+}{1-\beta(1-\lambda)} + \frac{\beta\lambda p^0}{1-\beta(1-\lambda)}\left[j_0 + j_1\mathcal{E}(n)\right], \\
j_1 &= \frac{\psi p \alpha n^{\alpha-1}}{1-\beta(1-\lambda)}. \qquad (37)
\end{aligned}
$$

Solving for j_0 we obtain the required solution for $J(n,x)$. Likewise, we can obtain recursions for the marginal effects of n and θ:

$$
\begin{aligned}
J_n(n,x) &= -\frac{1}{1-\beta(1-\lambda)}\frac{1-\alpha}{n}\psi p x \alpha n^{\alpha-1} + \frac{\beta\lambda}{1-\beta(1-\lambda)}\int_{R(n)}^{R_v(n)} J_n(n,x')\,dG, \\
J_p(n,x) &= \frac{1}{1-\beta(1-\lambda)}\psi x \alpha n^{\alpha-1} + \frac{\beta\lambda}{1-\beta(1-\lambda)}\int_{R(n)}^{R_v(n)} J_p(n,x')\,d\tilde{G}, \\
J_\theta(n,x) &= -\frac{\eta\beta c + \beta\lambda\frac{c}{q^2}q'(\theta)\int_{R_v(n)} dG}{1-\beta(1-\lambda)} + \frac{\beta\lambda}{1-\beta(1-\lambda)}\int_{R(n)}^{R_v(n)} J_\theta(n,x')\,dG. \qquad (38)
\end{aligned}
$$

Again using the method of undetermined coefficients, and noting that the Cobb Douglas matching function implies $q = \mu\theta^{-\phi} \implies \frac{c}{q^2}q'(\theta) = -\frac{c}{q}\frac{\phi}{\theta}$, yields the required solutions for J_n, J_p and J_θ. ■

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