

Ambiguity Aversion: Implications for the Uncovered Interest Rate Parity Puzzle

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WEB APPENDIX

C1. A general numerical solution procedure for the ambiguity aversion model

Lemma 2 stated for the special case of $\bar{\sigma}_V = 0$ then $s_t = a_1 \hat{x}_{t,t}^t + a_2 r_t$, where a_1, a_2 are the same analytical coefficients as in equations (15), that characterize the rational expectations case. For the case $\bar{\sigma}_V > 0$ a more general numerical procedure is required to recover the coefficients a_1, a_2 . The solution to the ambiguity aversion equilibrium can be summarized by the following steps:

1. Start with an initial guess about a_1, a_2 .
2. For each t , make a guess about the sign of b_t to use in (17).
3. Use (17) and call the resulting optimal sequence $\sigma_V^*(r^t)$. Use the Kalman filter based on the sequence $\sigma_V^*(r^t)$ to form an estimate for $\hat{x}_{t,t}^t$ and $\Sigma_{t,t}^t$.
4. Draw realizations for r_{t+1} from $N(\rho \hat{x}_{t,t}^t, \rho^2 \Sigma_{t,t}^t + \sigma_U^2)$, where $\Sigma_{t,t}^t$ is defined in (9b). Form the sample $r^{t+1} = (r^t, r_{t+1})$. For each realization perform Steps 2 and 3 above to obtain the sequence $\sigma_V^*(r^{t+1})$.
5. For each realization in step 4 use $\sigma_V^*(r^{t+1})$ to compute $\hat{x}_{t+1,t+1}^{t+1}$ and use the conjecture in (10) to generate a realized $s_{t+1} = a_1 \hat{x}_{t+1,t+1}^{t+1} + a_2 r_{t+1}$.
6. The distribution of s_{t+1} in step 5 defines the one step ahead probability distribution for the agent at time t . Use the FOC (19) to solve for s_t .
7. If $\text{sign}(s_t) = \text{sign}(b_t)$ the solution is $\sigma_V^*(r^t)$ and s_t and an indicator function $i_t = 1$. If $\text{sign}(s_t) \neq \text{sign}(b_t)$, switch the sign of the initial guess in step 2.
8. If there is no convergence on the sign of s_t and b_t , the solution is $b_t = s_t = 0$ and the indicator function $i_t = 0$.
9. Regress s_t on $\tilde{a}_1 \hat{x}_{t,t}^t$ and $\tilde{a}_2 r_t$ for all the t when $i_t = 1$. If $\min_{j \in \{1,2\}} |a_j - \tilde{a}_j| > \varepsilon$, then reiterate from step 1 with $a_j = \tilde{a}_j$. If not, then stop and the minimizing coefficients are \tilde{a}_j .

C2. Analytics of delayed overshooting

This section describes more formally the intuition of the delayed overshooting in section IV.D by analyzing the evolution of the estimate of the hidden state. At time t , the investor's estimate is $\hat{x}_{t,t}^t = \alpha K_t^t$, where

$$(C1) \quad K_t^t = \frac{\rho^2 \Sigma + \sigma_U^2}{\rho^2 \Sigma + \sigma_U^2 + \sigma_V^{2,H}},$$

where Σ is the steady state mean square error of the estimate, and $\Sigma_{t,t} = (1 - K_t^t)\sigma_U^2$. Her updated estimate at time $t + 1$ is:

$$(C2) \quad \begin{aligned} \hat{x}_{t+1,t+1}^{t+1} &= \rho\hat{x}_{t,t}^{t+1} + K_{t+1}^{t+1}(r_{t+1} - \rho\hat{x}_{t,t}^{t+1}) \\ K_{t+1}^{t+1} &= \frac{\rho^2\Sigma_{t,t} + \sigma_U^2}{\rho^2\Sigma_{t,t} + \sigma_U^2 + \sigma_V^{2,H}}. \end{aligned}$$

Because there are no further shocks in the true DGP after period t , then $r_{t+1} = \rho x_t$. Using $\bar{\sigma}_V = 0$ in the formula (28) for the true hidden state x_t , I get that $r_{t+1} = \rho\alpha$. The formula in (C2) can be first simplified by noting that $\hat{x}_{t,t}^{t+1} = \hat{x}_{t,t}^t$. Then, using the value of r_{t+1} and the value for $\hat{x}_{t,t}^t$ implied by (C1), the estimate in (C2) becomes:

$$(C3) \quad \hat{x}_{t+1,t+1}^{t+1} = \rho\alpha K_t^t + K_{t+1}^{t+1}(\rho\alpha - \rho\alpha K_t^t).$$

Notice that since $K_t^t < 1$, the investor observes a differential r_{t+1} that is higher than expected so that the perceived innovation $\rho\alpha - \rho\alpha K_t^t$ is positive. Compared to the RE case, this positive innovation will lead to an increase in the time $t + 1$ updated estimate of the hidden state. If this updating effect is strong enough, then $\hat{x}_{t+1,t+1}^{t+1}$ can be larger than $\hat{x}_{t,t}^t$. Then, since $s_t = a_1\hat{x}_{t,t} + a_2r_t$, where $a_1 < 0, a_2 < 0$, the exchange rate can appreciate between time t and $t + 1$, i.e. $s_{t+1} < s_t$. In that case, the currency experiences a delayed overshooting since it appreciates at time t and then continues to appreciate at time $t + 1$. The following condition on the parameters implies this delayed overshooting.

Condition 1: If $\rho(1 - K_t^t) \left(\frac{\rho}{1-\rho} K_{t+1}^{t+1} + 1 \right) > 1$ then $s_{t+1} < s_t < 0$.

Proof: The solution for the exchange rate is $s_t = a_1\hat{x}_{t,t} + a_2r_t$, where $a_1 = -\frac{\rho}{1-\rho}$ and $a_2 = -1$. We have $r_t = \alpha$ and $\hat{x}_{t,t}^t = \alpha K_t^t > 0$ so $s_t < 0$. The condition $s_{t+1} < s_t$ is then that

$$-\frac{\rho}{1-\rho}\hat{x}_{t+1,t+1}^{t+1} - r_{t+1} < -\frac{\rho}{1-\rho}\hat{x}_{t,t}^t - r_t.$$

Using the formula for $\hat{x}_{t+1,t+1}^{t+1}$ from (C3) and substituting in the values for r_t, r_{t+1} and $\hat{x}_{t,t}^t$ the above inequality reads:

$$-\frac{\rho}{1-\rho} [\rho\alpha K_t^t + K_{t+1}^{t+1}(\rho\alpha - \rho\alpha K_t^t)] - \rho\alpha < -\frac{\rho}{1-\rho}\alpha K_t^t - \alpha.$$

Rearranging, I get Condition 1.

It is easy to see that Condition 1 is satisfied when ρ is close to one and K_t^t is relatively small. Under the benchmark parameterization $K_t^t = 0.04$, $K_{t+1}^{t+1} = 0.07$, $\rho = 0.98$ and Condition 1 is easily satisfied. Notice that under RE, the condition

is naturally not met. In that case, assuming the steady state convergence on the Kalman gain, I get: $K_t = K_{t+1} = K$, where $K = \frac{\rho^2 \Sigma + \sigma_U^2}{\rho^2 \Sigma + \sigma_U^2 + \bar{\sigma}_V^2} = 1$.

C3. Risk aversion and distorted expectations

Consider a mean variance utility:

$$V_t = \max_{b_t} \min_{\tilde{\sigma}_V^{(t)}} E_t^{\tilde{P}} [b_t(s_{t+1} - s_t - r_t)] - \frac{1}{2} b_t^2 \text{Var}_t^{\tilde{P}} s_{t+1},$$

where the minimization is over the same sequence of variances as in (6). Suppose the equilibrium is characterized by a similar law of motion as the guess in (10) with the coefficients a_1, a_2 potentially different in this case. Then:

$$\text{Var}_t^{\tilde{P}} s_{t+1} = (a_1 K + a_2)^2 \text{Var}_t^{\tilde{P}} r_{t+1}.$$

By the Kalman filtering formulas, as in (13b), (13c), and Assumption 1, we have:

$$\text{Var}_t^{\tilde{P}} r_{t+1} = \rho^2 \Sigma_{t,t}^t + \sigma_U^2.$$

It is then easy to establish that:

PROPOSITION 4: *The variance of excess payoff, $b_t^2 \text{Var}_t^{\tilde{P}} s_{t+1}$, is increasing in $\sigma_{V,(t),t}^2$.*

Proof. Using the conjecture (10), Assumption 1 and taking as given $\hat{x}_{t,t}^{t+1}, K_{t+1}^{t+1}$:

$$\text{Var}_t^{\tilde{P}} s_{t+1} = (a_1 K + a_2)^2 [\rho^2 \Sigma_{t,t}^t + \sigma_U^2].$$

Use the formula in (10) for the Kalman gain and the recursion $\Sigma_{t,t}^t = \Sigma_{t,t-1}^t (1 - K_t)$. Then,

$$\begin{aligned} \frac{\partial \Sigma_{t,t}^t}{\partial \sigma_{V,(t),t}^2} &= \frac{\partial \Sigma_{t,t}^t}{\partial K_t} \frac{\partial K_t^t}{\partial \sigma_{V,(t),t}^2} > 0 \\ \frac{\partial (b_t^2 \text{Var}_t^{\tilde{P}} s_{t+1})}{\partial \sigma_{V,(t),t}^2} &> 0. \end{aligned}$$

This establishes *Proposition 4*.

Intuitively, a larger variance of the temporary shocks translates directly into a higher variance of the estimates $\Sigma_{t,t}^t$. By choosing higher values of σ_V in the

sequence $\tilde{\sigma}_V^{(t)}$, the agent will increase the expected variance of the differential $Var_t^{\tilde{P}} r_{t+1}$ because $\frac{\partial \Sigma_{t,t}^t}{\partial \sigma_{V,(t),t}^2} > 0$.

The overall effect of $\sigma_{V,(t),t}^2$ on V_t is then working through two channels. One is the positive relationship between $\sigma_{V,(t),t}^2$ and the variance of payoffs as in Proposition 4. The other effect, given by Corollary 1, is through expected payoffs. The total partial derivative is:

$$\frac{\partial V_t}{\partial \sigma_{V,(t),t}^2} = \frac{\partial V_t}{\partial E_t^{\tilde{P}} s_{t+1}} \frac{\partial E_t^{\tilde{P}} s_{t+1}}{\partial E_t^{\tilde{P}} r_{t+1}} \frac{\partial E_t^{\tilde{P}} r_{t+1}}{\partial \sigma_{V,(t),t}^2} + \frac{\partial V_t}{\partial Var_t^{\tilde{P}} s_{t+1}} \frac{\partial Var_t^{\tilde{P}} s_{t+1}}{\partial Var_t^{\tilde{P}} r_{t+1}} \frac{\partial Var_t^{\tilde{P}} r_{t+1}}{\partial \sigma_{V,(t),t}^2}.$$

Using Corollary 1 and Proposition 4, the sign of this derivative is:

$$sign \left[\frac{\partial V_t}{\partial \sigma_{V,(t),t}^2} \right] = sign(b_t) sign(r_t - \rho \hat{x}_{t-1,t-1}^t) - sign \left[\frac{\partial Var_t^{\tilde{P}} r_{t+1}}{\partial \sigma_{V,(t),t}^2} \right].$$

Since $sign\left(\frac{\partial Var_t^{\tilde{P}} r_{t+1}}{\partial \sigma_{V,(t),t}^2}\right) > 0$, if the sign of $sign(b_t) sign(r_t - \rho \hat{x}_{t-1,t-1}^t)$ is also positive then the sign of $\frac{\partial V_t}{\partial \sigma_{V,(t),t}^2}$ is ambiguous. To study that case, compute

$$\begin{aligned} \frac{\partial V_t}{\partial \sigma_{V,(t),t}^2} &= -(a_1 K + a_2) \rho b_t (r_t - \rho \hat{x}_{t-1,t-1}^t) \frac{\rho^2 \Sigma_{t-1,t-1}^t + \sigma_U^2}{[\rho^2 \Sigma_{t-1,t-1}^t + \sigma_U^2 + \sigma_{V,(t),t}^2]^2} \\ &\quad + (1 - \gamma)(a_1 K + a_2)^2 b_t^2 \rho^2 \frac{[\rho^2 \Sigma_{t-1,t-1}^t + \sigma_U^2]^2}{[\rho^2 \Sigma_{t-1,t-1}^t + \sigma_U^2 + \sigma_{V,(t),t}^2]^2}. \end{aligned}$$

By the filtering solution $(r_t - \rho \hat{x}_{t-1,t-1}^t) = (\rho^2 \Sigma_{t-1,t-1}^t + \sigma_U^2 + \sigma_{V,(t),t}^2)^{0.5} \xi_t$, where $\xi_t \sim N(0, 1)$. To investigate the ambiguous case compute the probability

$$(C4) \quad \Pr \left[\frac{\partial V_t}{\partial \sigma_{V,(t),t}^2} > 0 \mid (b_t \xi_t > 0) \right]$$

To get an upper bound on the probability take the case of $K = 1, \Sigma_{t-1,t-1}^t = 0$ and $\sigma_{V,(t),t}^2 = 0$. In this case, if $b_t > 0$ then (C4) becomes:

$$\Pr[\xi_t > (1 - \gamma)(a_1 + a_2) \rho b_t \sigma_U \mid \xi_t > 0].$$

Adding to the benchmark parameterization $\gamma = 10$ and noting that $b_t = 0.5s_t$ with the model-implied $\sqrt{Var_t^{\tilde{P}} s_{t+1}} = 0.025$ the probability that the expected return channel dominates is close to one. A similar calculation applies for $b_t < 0$.

C4. Time-varying parameters

Here I discuss a setup with time-varying parameters presented in Section IV.F.

$$\begin{aligned} r_t &= \rho_t r_{t-1} + \sigma_{V,t} v_t \\ \rho_t &= \rho_{t-1} + \sigma_{U,\rho} u_t, \end{aligned}$$

where r_t is the observable interest rate differential and ρ_t is a hidden parameter. The shocks u_t and v_t are white noise. The agent entertains the possibility that the $\sigma_{V,t}$ realizations are draws from the set $\Upsilon_\rho = \{\sigma_{V,\rho}^L, \bar{\sigma}_{V,\rho}, \sigma_{V,\rho}^H\}$.

Consider a similar framework for the two country general equilibrium model and assume risk neutrality. The UIP condition then states: $s_t = E_t^{\tilde{P}} s_{t+1} - r_t$ where \tilde{P} is the distorted belief. Solving forward the UIP condition implies that without time-variation or ambiguity the solution would simply be $s_t = \frac{1}{\rho-1} r_t$. Resorting to a model of anticipated utility (as in Sargent (1999)) makes agents ignore future updates about ρ in forecasting. Then the solution is:

$$s_t = (\hat{\rho}_t - 1)^{-1} r_t,$$

where $\hat{\rho}_t = E_t^{\tilde{P}}(\rho_t | I_t)$. Note that $E_t^{\tilde{P}} s_{t+1} = (\hat{\rho}_t - 1)^{-1} E_t^{\tilde{P}} r_{t+1}$ and $E_t^{\tilde{P}} r_{t+1} = \hat{\rho}_t r_t$. The expected return from investing in the foreign bond is $q_{t+1}^e = E_t^{\tilde{P}} s_{t+1} - s_t - r_t = (\hat{\rho}_t - 1)^{-1} \hat{\rho}_t r_t - r_t - s_t$. For a deterministic sequence $\tilde{\sigma}_V^{(t)}$, the Kalman filter delivers:

$$\begin{aligned} \hat{\rho}_t &= \hat{\rho}_{t-1} + K_t (r_t - \hat{\rho}_{t-1} r_{t-1}) \\ K_t &= \frac{\Sigma_{t-1} r_{t-1}}{r_{t-1}^2 \Sigma_{t-1} + \sigma_{V,t}^2} \\ \Sigma_t &= \Sigma_{t-1} - K_t \Sigma_{t-1} r_{t-1} + \sigma_{U,\rho}^2. \end{aligned}$$

Thus, given that q_{t+1}^e is increasing in $\hat{\rho}_t$, a similar result applies as in Corrolary 1.

CORROLARY 2: *The expected excess payoff, $E_t^{\tilde{P}} b_t q_{t+1}^e$, is monotonic in $\sigma_{V,(t),t}^2$. The monotonicity is given by the sign of $[b_t(r_t - \hat{\rho}_{t-1} r_{t-1})]$.*

The implication of Corrolary 2 is a decision rule similar to (17)

$$\begin{aligned} \sigma_{V,(t),t} &= \sigma_{V,\rho}^H \text{ if } b_t(r_t - \hat{\rho}_{t-1} r_{t-1}) < 0 \\ \sigma_{V,(t),t} &= \sigma_{V,\rho}^L \text{ if } b_t(r_t - \hat{\rho}_{t-1} r_{t-1}) > 0. \end{aligned}$$

Finally, the same market clearing condition as in (2) and the same equilibrium considerations as in section (III.B) would apply to this model.