Imperfect Competition in the Interbank Market
for Liquidity as a Rationale for Central Banking

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Web Appendix

Ex-ante liquidity insurance

We have considered liquidity transfers following a liquidity shock, ignoring the possibility for banks to insure against such shocks (e.g., Bhattacharya and Gale (1987), Allen and Gale (2000), Leitner (2005)). To consider this possibility, we modify the model as follows.

At $t = 0$, Bank A can seek liquidity insurance from Bank B. We assume Bank A cannot get liquidity insurance from outsiders. This seems consistent with banks being special in provision of lines of credit, not just to other borrowers, but also to each other.

At $t = 1$, Bank A's assets need funding of ρ with probability x, and no funding otherwise. Whether Bank A incurs a liquidity shock is verifiable. If Bank A incurs a liquidity shock, Bank B's opportunity cost of capital is non-verifiable and equal to $\mu > 1$ with probability y and 1 otherwise. We assume $p_H R < \mu\rho$, i.e., if Bank B's cost is high, transfers from Bank B to Bank A are inefficient. At that point, the banks can renegotiate.

We also make simplifying assumptions. First, we assume that all of Bank A's assets have the same characteristic θ.[1] Second, we assume that Bank A makes Bank B a take-it-or-leave it offer at $t = 0$. Third, we assume that Bank B cannot pledge any of its assets to Bank A. This ensures that if Bank B makes a transfer to Bank A but turns out to have a high cost of capital, Bank A will not transfer back the appropriate amount of liquidity to Bank B. Fourth, we assume that Bank B has full bargaining power in renegotiation. Finally, we

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[1] With heterogeneous values of θ, the optimal contract would generally involve some asset sale to Bank B even when Bank A does not incur a liquidity shock. Indeed, the sale of more liquid loans absent a shock could avoid the sale of less liquid loans in the event of a shock.

assume that outside markets are so weak that only Bank B can refinance Bank A's assets, i.e., any loan not refinanced by Bank B must be terminated.

In principle, for each of the three states $\omega \in \{(\rho, 1), (\rho, \mu), (0, 1)\}$, Bank A's offer specifies a transfer $T(\omega)$ from Bank B to Bank A, a set of assets of measure $\alpha(\omega)$ transferred to Bank B, and a claim $r(\omega)$ by Bank B on Bank A's remaining assets. However, that Bank B's cost of capital is unobservable constrains the set of feasible contracts at $t = 0$. It can be shown that the contract's terms cannot differ across states $(\rho, 1)$ and (ρ, μ).

PROPOSITION 7: *An optimal contract at $t = 0$ is as follows.[2] Define*

(A1)
$$T^* \equiv \frac{(1-x)\, p_H\, (R - R_b)\, \rho}{x \left[(1-y)\, (p_B R - p_H (R - R_b)) + y \frac{R_b}{R} \mu \rho \right]}.$$

- *If $T^* \geq \rho$, Bank A gets full liquidity insurance from Bank B so that no assets are sold to Bank B in the event of a liquidity shock, i.e., $T(\rho) = \rho$.*

- *If $T^* < \rho$, Bank A gets only partial liquidity insurance from Bank B, i.e.,*

  (A2)
  $$T(\rho) = T^*.$$

  If Bank A incurs a liquidity shock, Bank B acquires a fraction α^* of Bank A's assets with

  (A3)
  $$\alpha^* = 1 - T^*/\rho.$$

COROLLARY 4: *The fraction of assets Bank A sells after a liquidity shock, and the associated inefficiency increase with the probability x of a liquidity shock, the probability y that Bank B's cost of capital is high, and with the value μ of the high cost of capital.*

In other words, if a shock is more likely, there is less scope for liquidity insurance. As y or μ increases, Bank B is less keen to commit to a transfer to Bank A. As an implication, when an aggregate liquidity shortage is more likely, there is less insurance. In turn, even absent an aggregate liquidity shortage, Bank B can exploit its market power against Bank A. Finally, as long as liquidity insurance is only partial, the central bank can improve efficiency by acting as a lender of last resort.

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[2] This contract is not uniquely optimal. Indeed, a contract with some asset sales when there is no shock and less sales when there is a shock can also be optimal.

Proof of Proposition 1

Setting $r = R - R_b$ is weakly optimal because Bank A can always compensate an increase in r with an offsetting increase in T. Bank B's participation constraint is binding since, otherwise, Bank A can always increase T. Hence, we have

$$(22) \qquad T = p_H(R - R_b)F(\hat{\theta}) + \int_{\hat{\theta}}^{1} [p_B(\theta)R - \rho] \, dF(\theta) - X_B.$$

Substituting, we can write Bank A's problem as:

$$(23) \qquad \begin{aligned} \max_{\hat{\theta}} \quad & \int_0^{\hat{\theta}} p_H R dF(\theta) + \int_{\hat{\theta}}^1 p_B(\theta) R dF(\theta) - \rho - X_B \\ s.t. \quad & F(\hat{\theta})p_H(R - R_b) + \int_{\hat{\theta}}^1 [p_B(\theta)R - \rho] \, dF(\theta) - X_B - F(\hat{\theta})\rho \geq 0 \end{aligned}$$

As $p_H > p_B(\theta)$, the objective increases in $\hat{\theta}$. By condition (6), the constraint holds for $\hat{\theta} = 0$. Moreover, its LHS decreases with $\hat{\theta}$ (Assumption 1). If the constraint holds for $\hat{\theta} = 1$, i.e. if $p_H(R - R_b) - \rho \geq X_B$, then $\theta_A^* = 1$. If so, Bank A borrows more than needed to fund all its assets, i.e. $T_A^* > \rho$. This is equivalent to borrowing $T_A^* = \rho$ against a claim $r_A^* = (X_B + \rho)/p_H$. Otherwise, it is optimal to set θ_A^* so that it binds, i.e. $T_A^* = F(\theta_A^*)\rho$ and θ_A^* is as in (9).

Proof of Proposition 2

As before setting $r = R - R_b$ is weakly optimal. Since Bank A can always increase T, one of the other two constraints must bind. Hence Bank B's problem is

$$(24) \qquad \begin{aligned} \max_{\hat{\theta}, T} \quad & \int_0^{\hat{\theta}} p_H(R - R_b)dF(\theta) + \int_{\hat{\theta}}^1 [p_B(\theta)R - \rho] \, dF(\theta) - T \\ s.t. \quad & T = F(\hat{\theta})\rho + \max\left\{ E(\pi_A) - F(\hat{\theta})p_H R_b; 0 \right\}. \end{aligned}$$

If $E(\pi_A) > F(\hat{\theta})p_H R_b$, the objective becomes

$$(25) \qquad \int_0^{\hat{\theta}} p_H R dF(\theta) + \int_{\hat{\theta}}^1 p_B(\theta) R dF(\theta) - \rho - E(\pi_A),$$

which increases with $\hat{\theta}$, i.e. $F'(\hat{\theta})(p_H R - p_B(\hat{\theta})R) > 0$.

3

If $E\left(\pi_A\right) \geq p_H R_b F(\hat{\theta})$ for $\hat{\theta} = 1$, i.e. if $E\left(\pi_A\right) \geq p_H R_b$, then $\theta^* = 1$. This implies $T^* = \rho + E\left(\pi_A\right) - p_H R_b$. Bank A borrows more than needed to fund all its assets, i.e. $T^* > \rho$. This is equivalent to borrowing $T^* = \rho$ against a claim $r^* = R - E\left(\pi_A\right)/p_H$.

If $E\left(\pi_A\right) < p_H R_b$, it may be that $E\left(\pi_A\right) < p_H R_b F(\hat{\theta})$, in which case the objective is:

$$(26) \qquad \int_0^{\hat{\theta}} p_H(R - R_b)dF(\theta) + \int_{\hat{\theta}}^1 p_B(\theta)RdF(\theta) - \rho.$$

From Assumption 1, this objective decreases with $\hat{\theta}$, i.e. $(p_H(R - R_b) - p_B(\hat{\theta})R)F'(\hat{\theta}) < 0$. In that case, θ^* such that $E\left(\pi_A\right) = p_H R_b F(\theta^*)$ is optimal.

Proof of Corollary 1

$\alpha^* = 0$ if and only if $E\left(\pi_A\right) \geq p_H R_b$. (6) and (9) imply $\pi_A > X_A$. Hence, $\partial E\left(\pi_A\right)/\partial\beta < 0$. The condition holds for $\beta = 0$ if

$$(27) \qquad p_H(R - R_b) - \rho \geq X_B$$

and is violated for $\beta = 1$ if $\theta^* < 1$, i.e. if

$$(28) \qquad p_H R_b > X_A.$$

Note that under (6), (27) and/or (28) must hold. When both hold, $\beta^* \in (0,1)$. When only (27) holds, $\beta^* = 1$. When only (28) holds, $\beta^* = 0$. For $\beta > \beta^*$, $\theta^* = F^{-1}\left(E\left(\pi_A\right)/p_H R_b\right)$, which is strictly decreasing with $E\left(\pi_A\right)$, which is itself strictly decreasing with β.

Proof of Corollary 2

For $\beta^* \in (0,1)$, β^* is given by $E\left(\pi_A\right) = p_H R_b$, with $\pi_A = p_H R_b F\left(\theta_A^*\right)$. Hence

$$(29) \qquad \beta^* = \min\left\{1; \max\left\{0; 1 - \frac{p_H R_b - X_A}{p_H R_b F\left(\theta_A^*\right) - X_A}\right\}\right\}.$$

By inspection, $\partial\beta^*/\partial X_A < 0$. For $\beta^* \in (0,1)$, the denominator D's derivative with respect to X_B is

$$(30) \qquad \frac{\partial D}{\partial X_B} = \left(\frac{\partial\theta_A^*}{\partial X_B}\right)\left(\frac{\partial D}{\partial\theta_A^*}\right) = -\left(\frac{\partial\theta_A^*}{\partial X_B}\right)p_H R_b F'(\theta_A^*) > 0.$$

4

Similarly, if $\beta > \beta^*$, we have $E(\pi_A) = p_H R_b F(\theta^*)$, which can be rewritten as

$$p_H R_b F(\theta^*) = \beta X_A + (1 - \beta) p_H R_b F(\theta_A^*).$$

The LHS increases with θ^* while we have

$$\frac{\partial RHS}{\partial X_A} = \beta > 0 \qquad \frac{\partial RHS}{\partial \theta_A^*} = (1 - \beta) p_H R_b F'(\theta_A^*) > 0,$$

$$\frac{\partial RHS}{\partial \rho} = \left(\frac{\partial RHS}{\partial \theta_A^*}\right)\left(\frac{\partial \theta_A^*}{\partial \rho}\right) < 0, \quad \text{and} \quad \frac{\partial RHS}{\partial X_B} = \left(\frac{\partial RHS}{\partial \theta_A^*}\right)\left(\frac{\partial \theta_A^*}{\partial X_B}\right) < 0.$$

These, together with $\frac{\partial \alpha^*}{\partial \theta^*} < 0$ and $\frac{\partial K^*}{\partial \theta^*} < 0$, complete the proof.

Proof of Proposition 3

$\beta > \beta^*$ requires $p_H(R - R_b^o) < \rho$. Using (17), we have

$$(31) \qquad \frac{\partial \theta_o^*}{\partial R_b^o} = \frac{p_H F(\theta_o^*)}{-\left(p_o(\theta_o^*)R - p_H(R - R_b^o)\right) F'(\theta_o^*)},$$

which is negative because θ_o^* is the largest solution to (17). Moreover,

$$(32) \qquad \frac{\partial X_A}{\partial \theta_o^*} = (p_H - p_o(\theta_o^*))RF'(\theta_o^*) > 0.$$

Hence, $\frac{\partial X_A}{\partial R_b^o} < 0$, implying that $\frac{\partial X_A}{\partial b_o} < 0$. The implications for α^* and K^* stem from Corollary 2.

Consider now two distributions F_1 and F_2 with $F_1 < F_2$ over $(0, 1)$ such that $F = x F_1 + (1 - x) F_2$ with $x \in [0, 1)$. A shift of F towards higher values in the sense of FOSD corresponds to an increase in x. Bank A's outside option is $X_A = p_H R_b^o F(\theta_o^*)$ with θ_o^* given by (17). Integrating by parts, (17) becomes

(33)

$$p_o(1)R - p_H(R - R_b^o) - [p_o(\theta_o^*)R - p_H(R - R_b^o)] F(\theta_o^*) - \int_{\theta_o^*}^{1} \frac{\partial p_o}{\partial \theta}(\theta)RF(\theta)d\theta = \rho - p_H(R - R_b^o)$$

Taking the first derivative with respect to x yields

$$-\frac{\partial \theta_o^*}{\partial x}\frac{\partial p_o}{\partial \theta}(\theta_o^*)RF(\theta_o^*) - [p_o(\theta_o^*)R - p_H(R - R_b^o)]\frac{\partial}{\partial x}F(\theta_o^*)$$

$$-\int_{\theta_o^*}^{1} \frac{\partial p_o}{\partial \theta}(\theta)R[F_1(\theta) - F_2(\theta)]d\theta + \frac{\partial \theta_o^*}{\partial x}\frac{\partial p_o}{\partial \theta}(\theta_o^*)RF(\theta_o^*) = 0,$$

5

which given $\partial p_o/\partial\theta > 0$ and $F_1 < F_2$, implies

$$(34) \qquad \frac{\partial}{\partial x}F(\theta_o^*) = \frac{\int_{\theta_o^*}^1 \frac{\partial p_o}{\partial\theta}(\theta)R[F_2(\theta) - F_1(\theta)]d\theta}{p_o(\theta_o^*)R - p_H(R - R_b^o)} > 0.$$

Hence $\partial X_A/\partial x > 0$. Now turn to bargaining between banks A and B. If $\pi_A = p_H R - \rho - X_B$, $\partial\pi_A/\partial x = 0$. Otherwise, $\pi_A = p_H R_b F(\theta_A^*)$ with θ_A^* given by (9). Similar steps yield

$$(35) \qquad \frac{\partial}{\partial x}F(\theta_A^*) = \frac{\int_{\theta_A^*}^1 \frac{\partial p_B}{\partial\theta}(\theta)R[F_2(\theta) - F_1(\theta)]d\theta}{p_B(\theta_A^*)R - p_H(R - R_b)} > 0.$$

Hence $\partial\pi_A/\partial x > 0$, which together with $\partial X_A/\partial x > 0$ implies $\partial E(\pi_A)/\partial x > 0$. The fraction of assets sold, $\alpha^* = (1 - E(\pi_A)/p_H R_b)$, decreases with x as does the threshold $\theta^* = F^{-1}(E(\pi_A)/p_H R_b)$. Integrating by parts, the resulting inefficiency can be written as

$$K^* = (p_H - p_B(1)) R - (p_H - p_B(\theta^*)) RF(\theta^*) + \int_{\theta^*}^1 \frac{\partial p_B}{\partial\theta}(\theta)RF(\theta)d\theta.$$

Noting that $p_H > p_B(\theta^*)$, $\frac{\partial}{\partial x}(E(\pi_A)/p_H R_b) > 0$, $\frac{\partial p_B}{\partial\theta} > 0$ and $F_2 > F_1$ implies

$$\begin{aligned}
\frac{\partial K^*}{\partial x} &= \frac{\partial\theta^*}{\partial x}\frac{\partial p_B}{\partial\theta}(\theta^*)RF(\theta^*) - (p_H - p_B(\theta^*)) R\frac{\partial}{\partial x}(E(\pi_A)/p_H R_b)\\
&\quad + \int_{\theta^*}^1 \frac{\partial p_B}{\partial\theta}(\theta)R(F_1(\theta) - F_2(\theta))d\theta - \frac{\partial\theta^*}{\partial x}\frac{\partial p_B}{\partial\theta}(\theta^*)RF(\theta^*)\\
&= -(p_H - p_B(\theta^*)) R\frac{\partial}{\partial x}(E(\pi_A)/p_H R_b) - \int_{\theta^*}^1 \frac{\partial p_B}{\partial\theta}(\theta)R(F_2(\theta) - F_1(\theta))d\theta < 0.
\end{aligned}$$

Proof of Proposition 4

Say the central bank lends $L_C = p_H(R - R_b^C)F(\theta_C)$ against assets with $\theta \in [0, \theta_C]$, and makes a transfer T_C. Then, if needed, Bank A borrow from outsiders and sell them assets with $\theta \in [\theta_o^*, 1]$. We can apply Lemma 1, replacing ρ with $(\rho - T_C - L_C)$ and assuming the measure of asset is $(1 - F(\theta_C))$, not 1. Hence $\theta_o^* = 1$ if

$$(36) \qquad L_C + (1 - F(\theta_C))p_H(R - R_b^o) \geq \rho - T_C.$$

Otherwise, θ_o^* is the largest solution to

$$(37) \qquad L_C + [F(\theta_o^*) - F(\theta_C)]p_H(R - R_b^o) + \int_{\theta_o^*}^{1} p_o(\theta)RdF(\theta) = \rho - T_C.$$

The central bank maximizes θ_o^* subject to $T_C \in [0, \Lambda]$.

Case 1. $b_C \geq b_o$ ($\Leftrightarrow R_b^C \geq R_b^o$): (36)'s LHS is maximal for $\theta_C = 0$ and equals $p_H(R - R_b^o) < \rho$.

Case 1.1. $p_H(R - R_b^o) \geq \rho - \Lambda$: As the central bank can achieve $\theta_o^* = 1$, it minimizes T_C subject to $\theta_o^* = 1$, i.e. to (36) and $T_C \geq 0$. Since $p_H(R - R_b^o) < \rho$, the latter constraint is slack, i.e. $T_C > 0$. Hence the central bank makes a transfer $T_C = \rho - p_H(R - R_b^o)$ and no loan (i.e. $L_C = 0$).

Case 1.2. $p_H(R - R_b^o) < \rho - \Lambda$: As the central bank cannot achieve $\theta_o^* = 1$, it maximizes θ_o^* subject to (37), $T \in [0, \Lambda]$ and $\theta_C \in [0, 1]$. As (36)'s maximum obtains for $\theta_C = 0$ and $T_C = \Lambda$, this also maximizes θ_o^*.

Case 2. $b_C < b_o$ ($\Leftrightarrow R_b^C < R_b^o$). (36)'s LHS is maximal for $\theta_C = 1$ and equals $p_H(R - R_b^C)$.

Case 2.1. $p_H(R - R_b^C) \geq \rho - \Lambda$: As the central bank can achieve $\theta_o^* = 1$, it minimizes T_C subject to $\theta_o^* = 1$, i.e. to (36) and $T_C \geq 0$.

Case 2.1.1. $p_H(R - R_b^C) \geq \rho$: $T_C \geq 0$ binds, i.e. $T_C = 0$. Hence the central bank makes a loan $L_C \geq \rho$ but no transfer (i.e. $T_C = 0$). The largest loan, $L_C = p_H(R - R_b^C)$, is weakly optimal.

Case 2.1.2. $p_H(R - R_b^C) < \rho$: $T_C \geq 0$ is slack, i.e. $T_C > 0$. Hence the central bank maximizes L_C (i.e. $L_C = p_H(R - R_b^C)$) and makes a transfer $T_C = \rho - p_H(R - R_b^o)$.

Case 2.2. $p_H(R - R_b^C) < \rho - \Lambda$: As the central bank cannot achieve $\theta_o^* = 1$, it maximizes θ_o^* subject to (37), $T \in [0, \Lambda]$ and $\theta_C \in [0, 1]$. As (36)'s LHS is maximal for $\theta_C = \theta_o^*$ and $T_C = \Lambda$, this also maximizes θ_o^*.

Proof of Propositions 5 and 6

Since $\beta > \beta_o^*$, $p_H(R - R_b^o) < \rho$. Suppose $p_H(R - R_b^C) < \rho$ and denote $R_b^m \equiv \min\{R_b^o, R_b^C\}$. Say the central bank sets $\Lambda \in [0, \bar{\Lambda}]$ with $p_H(R - R_b^m) < \rho - \Lambda$. (Below we show this is weakly

optimal). From Proposition 4, ex post the central bank would make a transfer $T_C = \Lambda$ and outsiders would buy loans with $\theta < \theta_C^*$ given by

$$(38) \qquad p_H(R - R_b^m)F(\theta_C^*) + \int_{\theta_C^*}^{1} p_o(\theta)RdF(\theta) = \rho - \Lambda,$$

and Bank A's payoff would be $X_A = p_H R_b^m F(\theta_C^*)$.

Consider $b_C \geq b_o$ and $\Lambda = 0$. Since $R_b^m = R_b^o$, (38) coincides with (17). Hence Bank A's payoff, $X_A = p_H R_b^o F(\theta_o^*)$, is the same as absent the central bank. Hence the outcome of bargaining with Bank B is unchanged. This proves Proposition 5.

Consider $b_C < b_o$, $R_b^m = R_b^C$. Applying Proposition 3 replacing R_b^o with R_b^C, we get $\frac{\partial \alpha^*}{\partial b_C} > 0$ and $\frac{\partial K^*}{\partial b_C} > 0$.

Consider $\Lambda > 0$. Applying Proposition 3 replacing ρ with $\rho - \Lambda$ and R_b^o with R_b^C, we get $\frac{\partial \alpha^*}{\partial \Lambda} < 0$ and $\frac{\partial K^*}{\partial \Lambda} < 0$.

Suppose now that $p_H \left(R - R_b^C \right) \geq \rho$, which requires $b_C < b_o$. Ex post the central bank would make a loan $L_C \geq \rho$ and no transfer (Proposition 4). Hence Bank A's payoff would be $X_A = p_H R - \rho$, implying that bargaining with Bank B yields the efficient outcome $\alpha^* = 0$.

Proof of Proposition 7

Consider $\omega = (\rho, 1)$. Following the transfer $T(\omega)$, Bank A can fund a fraction $T(\omega)/\rho$ of its assets and hence, its expected payoff is $\pi_A(\omega) = p_H(R - r(\omega))T(\omega)/\rho$. Hence Bank B's best renegotiation offer ensures Bank A that same payoff, minimizing the fraction α' of assets sold to Bank B, i.e., $(1 - \alpha')p_H R_b = \pi_A(\omega)$. (Note that since $r(\omega) \leq R - R_b$, we have $(1 - \alpha') \geq T(\omega)/\rho$, i.e., Bank B does not decrease its transfer to Bank A.) Hence Bank B's expected payoff is

$$\begin{aligned} \pi_B(\omega) &= \alpha' p_B R + (1 - \alpha')p_H(R - R_b) - \rho & (A4)\\ &= \left(1 - \frac{(R - r(\omega))T(\omega)}{R_b \rho}\right)p_B R + \frac{(R - r(\omega))T(\omega)}{R_b \rho}p_H(R - R_b) - \rho & (A5) \end{aligned}$$

Consider $\omega = (\rho, \mu)$. Bank A's expected payoff is $\pi_A(\omega) = p_H(R - r(\omega))T(\omega)/\rho$. Bank B's best renegotiation offer ensures Bank A that same payoff, minimizing transfer T', which amounts to minimizing the fraction $(1 - \alpha')$ of assets retained by Bank A, i.e.,

8

$(1 - \alpha') p_H R = \pi_A(\omega)$. (Note that since $r(\omega) \geq 0$, we have $(1 - \alpha') \leq T(\omega)/\rho$, i.e., Bank B does not increase its transfer to Bank A.) Hence Bank B's expected payoff is

$$\pi_B(\omega) = -(1 - \alpha') \rho \mu = -\frac{(R - r(\omega)) T(\omega)}{R_b} \mu.$$

Consider $\omega = (0, 1)$. It is easily seen that the maximum expected payoff the contract can ensure without asset sales is $\pi_B(\omega) = p_H(R - R_b)$.

If there is no contract at $t = 0$, Bank B's payoff is zero in all states except $\omega = (\rho, 1)$ in which it can acquire all of Bank A's assets for no transfer and refinance them, so that its expected payoff is $\pi_B = x(1 - y)(p_B R - \rho)$.

The optimal contract chosen by Bank A at $t = 0$ maximizes $T(\rho)$ subject to

$$(1 - x) \pi_B(0, 1) + x(1 - y) \pi_B(\rho, 1) + xy \pi_B(\rho, \mu) \geq \pi_B.$$

This can be rewritten as

$$T(\rho) \leq \frac{(1 - x) p_H(R - R_b) \rho}{x(R - r(\rho)) \left[(1 - y) \frac{p_B R - p_H(R - R_b)}{R_b} + y \frac{\mu}{R} \rho\right]}.$$

The constraint is relaxed when $r(\rho)$ is maximized. Hence it is optimal to set $r(\rho) = (R - R_b)$. Given this, the constraint can be rewritten as $T(\rho) \leq T^*$ with T^* as in (A1).