Welfare-based optimal monetary policy with unemployment and sticky prices: A linear-quadratic framework: Appendix

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1 The efficient equilibrium

Using the assumed functional form of the matching function, the social planner's problem can be written as

$$\max_{C,\, N,\, u,\, \theta} E_t \sum_{i=0}^{\infty} \beta^i \left\{ \left(\frac{C_{t+i}^{1-\sigma}}{1-\sigma} \right) + \lambda_{t+i} \left[Z_{t+i} N_{t+i} - \kappa u_{t+i} \theta_{t+i} + w^u (1 - N_t) - C_{t+i} \right] \right.$$
$$\left. + \psi_{t+i} \left[(1-\rho) N_{t+i-1} + \varphi \theta_{t+i}^{\alpha} u_{t+i} - N_{t+i} \right] + s_{t+i} \left[u_{t+i} - 1 + (1-\rho) N_{t+i-1} \right] \right\}.$$

First order conditions are

$$C\colon\ C_t^{-\sigma} - \lambda_t = 0;$$

$$\theta\colon\ -\lambda_t \kappa u_{t+i} + \psi_t \alpha \varphi \theta_t^{\alpha-1} u_t = 0;$$

$$u\colon\ -\lambda_t \kappa \theta_t + \psi_t \varphi \theta_t^{\alpha} + s_t = 0;$$

$$N\colon\ \lambda_t (Z_t - w^u) - \psi_t + (1-\rho)\beta E_t \left(\psi_{t+1} + s_{t+1} \right) = 0.$$

The second of these first order conditions implies

$$\frac{\psi_t}{\lambda_t} = \left(\frac{\kappa}{\alpha\varphi} \theta_t^{1-\alpha} \right),$$

while the third then implies

$$\frac{s_t}{\lambda_t} = \kappa\theta_t - \frac{\psi_t}{\lambda_t} \varphi \theta_t^{\alpha} = \left(\frac{\alpha - 1}{\alpha} \right) \kappa \theta_t.$$

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We can write the fourth of these first order conditions as

$$\begin{aligned} Z_t &= w^u + \frac{\psi_t}{\lambda_t} - (1-\rho)\beta E_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\left(\frac{\psi_{t+1}+s_{t+1}}{\lambda_{t+1}}\right) \\ &= w^u + \left(\frac{\kappa}{\alpha\varphi}\theta_t^{1-\alpha}\right) - (1-\rho)\beta E_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\left[\frac{\kappa}{\alpha\varphi}\theta_{t+1}^{1-\alpha} + \left(\frac{\alpha-1}{\alpha}\right)\kappa\theta_{t+1}\right]. \end{aligned}$$

Rearranging this condition for efficiency and noting that $\varphi\theta_t^{\alpha-1} = q(\theta_t)$ yields

$$\begin{aligned} Z_t &= w^u + \frac{\kappa}{\alpha q(\theta_t)} - (1-\rho)\beta E_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\frac{\kappa}{\alpha q(\theta_{t+1})} \\ &\quad +(1-\rho)\left(\frac{1-\alpha}{\alpha}\right)\beta E_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\kappa\theta_{t+1}. \end{aligned} \tag{1}$$

In the steady-state, this condition becomes

$$1 = w^u + \frac{\kappa}{\alpha\bar{q}} - (1-\rho)\beta\frac{\kappa}{\alpha\bar{q}} + (1-\rho)\left(\frac{1-\alpha}{\alpha}\right)\beta\kappa\bar{\theta}$$

But as $\bar{q} = \rho\bar{N}/\bar{V}$ and $\bar{\theta} = \bar{V}/\bar{U}$, we have

$$\delta_1 \equiv 1 - w^u - \frac{\kappa\bar{V}}{\alpha\rho\bar{N}} = -\beta(1-\rho)\frac{\kappa\bar{V}}{\alpha\bar{U}}\left(\alpha - 1 + \frac{\bar{U}}{\rho\bar{N}}\right) \equiv -\beta\delta_2. \tag{2}$$

Note that δ_1 is the net consumption generated by one additional match (the match produces 1 unit, but home production falls by w^u, and to fill the march required posting $1/(\partial m/\partial v) = \bar{V}/\alpha\rho\bar{N}$ vacancies at a cost κ each).

Equations (11) and (12) imply that the job posting condition takes the form

$$\frac{Z_t}{\mu_t} = w^u + \left(\frac{1}{1-b_t}\right)\left\{\frac{\kappa}{q_t} - \beta(1-\rho)E_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)(1-b_t p_{t+1})\frac{\kappa}{q_{t+1}}\right\}.$$

When prices are flexible, $\mu_t = \mu > 1$ for all t. However, a tax-subsidy policy that offsets the allocative effects of the steady-state markup is not sufficient to ensure that the resulting flex-price equilibrium is efficient as is the case in the standard new Keynesian model. Inefficient job posting can lead to an inefficient level of vacancies and unemployment. Efficiency requires $\mu_t = \mu = 1$ *and* $b_t = 1-\alpha$. This second condition is the familiar the Hosios condition for efficient vacancy creation.[1] Letting a superscript e denote the efficient equilibrium, the job posting condition takes the form

$$Z_t = w^u + \left(\frac{1}{\alpha}\right)\left\{\frac{\kappa}{\varphi}(\theta_t^e)^{1-\alpha} - \beta(1-\rho)E_t\left(\frac{\lambda_{t+1}^e}{\lambda_t^e}\right)\kappa\left[\frac{\left(\theta_{t+1}^e\right)^{1-\alpha}}{\varphi} - (1-\alpha)\theta_{t+1}^e\right]\right\}.$$

[1] The Hosios condition requres that labor's share of the surplus, b, equal the elasticity of matches with respect to unemployment, $1-\alpha$.

When linearized,and expressed in terms of the efficient level of unemployment, this yields

$$a_1\left(\hat{u}^e_{t+1} - \rho_u \hat{u}^e_t\right) - \beta a_2\left(E_t \hat{u}^e_{t+2} - \rho_u \hat{u}^e_{t+1}\right) - \beta a_3 \alpha\rho\eta \hat{r}^e_t = -\alpha\rho\eta\hat{z}_t, \tag{3}$$

where r^e_t is the equilibrium real interest rate in the efficient equilibrium.

In addition, the IS relationship in the efficient equilibrium takes the form

$$\begin{aligned}\hat{u}^e_{t+1} &= \left(\frac{\beta}{1+\beta}\right) E_t \hat{u}^e_{t+2} + \left(\frac{1}{1+\beta}\right)\hat{u}^e_t \\ &\quad - \left(\frac{1}{\sigma}\right)\left(\frac{1}{1+\beta}\right)\left(\frac{\eta\bar{C}}{\delta_2 \bar{Y}}\right) r^e_t - \left(\frac{1}{1+\beta}\right)\left(\frac{\bar{Y}}{\bar{C}}\right)(1-\rho_z)\,\hat{z}_t. \end{aligned} \tag{4}$$

where $\delta_2 = (1-\rho)\left(\kappa\bar{V}/\alpha\bar{U}\right)\left(\alpha - 1 + \bar{U}/\rho\bar{N}\right)$.. The responses of the efficient unemployment rate $\hat{u}^e_{t+1}$ and the real interest rate in the face of productivity shocks can be found by jointly solving (3) and (4).[2] Equations (3) and (4) imply that the equilibrium unemployment responds to productivity shocks even under flexible prices.

2 The Nash bargaining solution

The value of a match to a firm is J_t. For a worker, the valuation equation for being in a match that produces in period t isFor a worker, the valuation equation for being in a match that produces in period t is

$$V^E_t = w_t + \beta \mathrm{E}_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\left\{(1-\rho)V^E_{t+1} + \rho\left[p_{t+1}V^E_{t+1} + (1-p_{t+1})V^U_{t+1}\right]\right\},$$

or

$$V^E_t = w_t + \beta \mathrm{E}_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\left[V^E_{t+1} - \rho\,(1-p_{t+1})\left(V^E_{t+1} - V^U_{t+1}\right)\right]$$

since a matched worker survives the exogenous separation hazard with probability $1-\rho$, is exogenously separated but finds another match with probability ρp_{t+1}, and fails to find a match with probability $1-p_{t+1}$. The valuation equation for being unmatched is

$$V^U_t = w^u + \beta \mathrm{E}_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\left[p_{t+1}(1-\rho)V^E_{t+1} + p_{t+1}\rho V^U_{t+1} + (1-p_{t+1})V^U_{t+1}\right],$$

or

$$V^U_t = w^u + \beta \mathrm{E}_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\left[V^U_{t+1} + p_{t+1}\left(V^E_{t+1} - V^U_{t+1}\right)\right],$$

[2] In (4), we have followed Neiss and Nelson (2003) in defining $\hat{u}^e_{t+1}$ relative to last-period's efficient unemployment rate. Thus, the path of $\hat{u}^e_{t+i}$ is that for an economy that has always been in an efficient equilibrium. Alternatively, Woodford (2003) defines the flex-price and efficient equilibria conditional on the actual outcomes in the previous period. In that case, $\hat{u}^e_{t+1}$ would depend on $\hat{u}_t$. Edge (2003) discusses these two alternative definitions in the context of a model in which the lagged capital stock is an endogenous state variable. We follow the Neiss-Nelson definition; as Edge shows, it proves more convenient for deriving the welfare approximation we use to characterize optimal monetary policy.

An unmatched worker finds a job and survives the exogenous separation hazard with probability $p_t(1-\rho)$, finds a job but does not survive the exogenous hazard with probability $q_t\rho$, and fails to find a job with probability $1-p_t$. Thus, the value of a match to worker is

$$V_t^S \equiv V_t^E - V_t^U = w_t - w^u + \beta(1-\rho)\mathrm{E}_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)(1-p_{t+1})\,V_{t+1}^S. \tag{5}$$

The value of a match to a worker is equal to the wage w_t net of w^u plus the expected discounted value of being in a match in the following period. The probability of being in a match is $1-\rho$, the probability of surviving the exogenous separation hazard, plus the probability of exogenously separating but finding another match within the period, or ρ.

Let b_t denote the worker's share of the job surplus in period t, where b_t is assumed to follow a stationary stochastic process. Under Nash bargaining, the sharing rule implies

$$b_t J_t = b_t\left(\frac{\kappa}{q_t}\right) = (1-b_t)\left(V_t^E - V_t^U\right). \tag{6}$$

so

$$V_{t+1}^S = \left(\frac{b_{t+1}}{1-b_{t+1}}\right)J_{t+1} = \left(\frac{b_{t+1}}{1-b_{t+1}}\right)\left(\frac{\kappa}{q_{t+1}}\right)$$

We can use this in (5) to write the worker's surplus as

$$\begin{aligned} V_t^S &\equiv V_t^E - V_t^U = w_t - w^u + \beta(1-\rho)\mathrm{E}_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)(1-p_{t+1})\,V_{t+1}^S \\ &= w_t - w^u + \beta(1-\rho)\mathrm{E}_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)(1-p_{t+1})\left(\frac{b_{t+1}}{1-b_{t+1}}\right)\left(\frac{\kappa}{q_{t+1}}\right). \end{aligned}$$

So (6) becomes

$$b_t\left(\frac{\kappa}{q_t}\right) = (1-b_t)V_t^S = (1-b_t)\left[w_t - w^u + \beta(1-\rho)\mathrm{E}_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)(1-p_{t+1})\left(\frac{b_{t+1}}{1-b_{t+1}}\right)\left(\frac{\kappa}{q_{t+1}}\right)\right]$$

Solving for the wage,

$$\begin{aligned} \left[w_t - w^u + \beta(1-\rho)\mathrm{E}_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)(1-p_{t+1})\left(\frac{b_{t+1}}{1-b_{t+1}}\right)\left(\frac{\kappa}{q_{t+1}}\right)\right] &= \left(\frac{b_t}{1-b_t}\right)\left(\frac{\kappa}{q_t}\right) \\ w_t = w^u + \left(\frac{b_t}{1-b_t}\right)\left(\frac{\kappa}{q_t}\right) - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\,(1-p_{t+1})\left(\frac{b_{t+1}}{1-b_{t+1}}\right)\left(\frac{\kappa}{q_{t+1}}\right) \end{aligned}$$

or, defining

$$\varphi_t = \left(\frac{b_t}{1-b_t}\right)\left(\frac{\kappa}{q_t}\right),$$

$$w_t = w^u + \varphi_t - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\,(1-p_{t+1})\,\varphi_{t+1} \tag{7}$$

Doing it correctly, equation (13) of the most recent draft is

$$\frac{Z_t}{\mu_t} = w_t + \left(\frac{\kappa}{q_t}\right) - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\left(\frac{\kappa}{q_{t+1}}\right)$$

which, using the wage equation (7) yields

$$\begin{aligned}\frac{Z_t}{\mu_t} &= w^u + \varphi_t - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\left(1-p_{t+1}\right)\varphi_{t+1} \\ &\quad + \left(\frac{\kappa}{q_t}\right) - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\left(\frac{\kappa}{q_{t+1}}\right)\end{aligned}$$

or

$$\begin{aligned}\frac{Z_t}{\mu_t} &= w^u + \left(\frac{b_t}{1-b_t}\right)\left(\frac{\kappa}{q_t}\right) - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\left(1-p_{t+1}\right)\left(\frac{b_{t+1}}{1-b_{t+1}}\right)\left(\frac{\kappa}{q_{t+1}}\right) \\ &\quad + \left(\frac{\kappa}{q_t}\right) - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\left(\frac{\kappa}{q_{t+1}}\right)\end{aligned}$$

Combining terms,

$$\frac{Z_t}{\mu_t} = w^u + \left(\frac{1}{1-b_t}\right)\left(\frac{\kappa}{q_t}\right) - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\left(\frac{1-p_{t+1}b_{t+1}}{1-b_{t+1}}\right)\left(\frac{\kappa}{q_{t+1}}\right)$$

So the definition of the ξ_t, the effective cost of labor, is

$$\xi_t \equiv w^u + \left(\frac{1}{1-b_t}\right)\left\{\left(\frac{\kappa}{q_t}\right) - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\left(\frac{1-b_t}{1-b_{t+1}}\right)\left(1-b_{t+1}p_{t+1}\right)\left(\frac{\kappa}{q_{t+1}}\right)\right\}$$

3 The linearized structural equations

The model consists of the following core equations (see paper for definitions of the variables and a discussion of the model equations):

$$Y_t = C_t - w^u\left(1-N_t\right) + \kappa V_t, \tag{8}$$

$$\lambda_t = \beta E_t R_t \lambda_{t+1}$$

$$u_t \equiv 1 - (1-\rho)\,N_{t-1}.$$

$$N_t = (1-\rho)\,N_{t-1} + \chi v_t^{\alpha} u_t^{1-\alpha}.$$

$$\frac{Z_t}{\mu_t} = w_t + \frac{\kappa}{q_t} - (1-\rho)\beta E_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\left(\frac{\kappa}{q_{t+1}}\right) \tag{9}$$

$$w_t = (1-b_t)w^u + b_t\left[\frac{Z_t}{\mu_t} + (1-\rho)\left(\frac{1}{R_t}\right)E_t\left(\frac{\kappa}{q_{t+1}}\right)p_{t+1}\right], \tag{10}$$

$$\tau_t \equiv w^u + \left(\frac{1}{1-b_t}\right)\left\{\frac{\kappa}{q_t} - (1-\rho)\left(\frac{1}{R_t}\right)E_t\left(1-b_t p_{t+1}\right)\frac{\kappa}{q_{t+1}}\right\}. \tag{11}$$

plus the Fisher equation linking the nominal and real interest rates, the first order condition for price adjusting firms, and the definitions of λ as the marginal utility of consumption, q_t as the probability a vacancy is filled, p_t as the probability an unemployment work finds a match, and b_t and z_t are exogenous bargaining and productivity shocks.

When linearized around the steady state of the model, one obtains

$$\hat{y}_t = \left(\frac{C}{Y}\right)\hat{c}_t + w^u\hat{n}_t + \left(\frac{\kappa V}{Y}\right)\hat{v}_t. \tag{12}$$

$$\hat{c}_t = E_t\hat{c}_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - E_t\pi_{t+1}\right). \tag{13}$$

$$\hat{u}_t = -(1-\rho)\left(\frac{\bar{N}}{\bar{u}}\right)\hat{n}_{t-1} \equiv -\eta\hat{n}_{t-1}. \tag{14}$$

$$\hat{n}_t = \rho_u\hat{n}_{t-1} + \alpha\rho\hat{\theta}_t, \tag{15}$$

where

$$\hat{\theta}_t = \hat{v}_t - \hat{u}_t. \tag{16}$$

Since for marginal cost,

$$\hat{\mu}_t = z_t - \hat{\xi}_t,$$

we need an expression for $\hat{\xi}_t$, the effective cost of labor. Start with

$$\xi_t \equiv w^u + \left(\frac{1}{1-b_t}\right)\left(\frac{\kappa}{q_t}\right) - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\left(\frac{1-b_{t+1}p_{t+1}}{1-b_{t+1}}\right)\left(\frac{\kappa}{q_{t+1}}\right)$$

The terms on the right hand side are approximated as

$$(1)\ \left(\frac{1}{1-b_t}\right)\left(\frac{\kappa}{q_t}\right) \approx \left(\frac{1}{1-b}\right)\left(\frac{\kappa}{q}\right) + \left(\frac{1}{1-b}\right)\left(\frac{\kappa}{q}\right)\left(\frac{b}{1-b}\right)\hat{b}_t - \left(\frac{1}{1-b}\right)\left(\frac{\kappa}{q}\right)\hat{q}_t$$

$$\begin{aligned}(2) - (1-\rho)\left(\frac{1}{R_t}\right)\mathrm{E}_t\left(\frac{1-b_{t+1}p_{t+1}}{1-b_{t+1}}\right)\left(\frac{\kappa}{q_{t+1}}\right) \quad = \quad & -\beta(1-\rho)\left(\frac{1-bp}{1-b}\right)\left(\frac{\kappa}{q}\right) \\ & -\beta(1-\rho)\left(\frac{1-bp}{1-b}\right)\left(\frac{\kappa}{q}\right)\hat{q}_{t+1} \\ & +\beta(1-\rho)\left(\frac{\kappa}{q}\right)\left(\frac{1-bp}{1-b}\right)\hat{r}_t \\ & -\beta(1-\rho)\left(\frac{\kappa}{q}\right)\mathrm{E}_t\left(\frac{1-b_{t+1}p_{t+1}}{1-b_{t+1}}\right)\end{aligned}$$

and

$$\left(\frac{1-b_{t+1}p_{t+1}}{1-b_{t+1}}\right) \approx \left(\frac{1-bp}{1-b}\right) + \left(\frac{1-p}{1-b}\right)\left(\frac{b}{1-b}\right)\hat{b}_{t+1} - \left(\frac{bp}{1-b}\right)\hat{p}_{t+1}$$

So collecting these results after noting that $\xi = 1/\mu$:

$$\begin{aligned}
\xi\hat{\xi}_t &= \left(\frac{1}{1-b}\right)\left(\frac{\kappa}{q}\right)\left(\frac{b}{1-b}\right)\hat{b}_t - \left(\frac{1}{1-b}\right)\left(\frac{\kappa}{q}\right)\hat{q}_t \\
&\quad -\beta(1-\rho)\left(\frac{1-bp}{1-b}\right)\left(\frac{\kappa}{q}\right)\hat{q}_{t+1} + \beta(1-\rho)\left(\frac{\kappa}{q}\right)\left(\frac{1-bp}{1-b}\right)\hat{r}_t \\
&\quad -\beta(1-\rho)\left(\frac{\kappa}{q}\right)\mathrm{E}_t\left[\left(\frac{1-p}{1-b}\right)\left(\frac{b}{1-b}\right)\hat{b}_{t+1} - \left(\frac{bp}{1-b}\right)\hat{p}_{t+1}\right]
\end{aligned}$$

Recall that

$$\hat{q}_t = (\alpha-1)\hat{\theta}_t;\ \hat{p}_t = \hat{\theta}_t + \hat{q}_t = \alpha\hat{\theta}_t$$

So

$$\begin{aligned}
\hat{\xi}_t &= -\mu\left(\frac{1}{1-b}\right)\left(\frac{\kappa}{q}\right)\hat{q}_t + \\
&\quad \mu\left(\frac{1}{1-b}\right)\left(\frac{\kappa}{q}\right)\left(\frac{b}{1-b}\right)\left[1-\beta(1-\rho)(1-p)\rho_b\right]\hat{b}_t \\
&\quad -\mu\beta(1-\rho)\left(\frac{\kappa}{q}\right)\mathrm{E}_t\left[\left(\frac{1-bp}{1-b}\right)(\alpha-1) - \left(\frac{bp}{1-b}\right)\alpha\right]\hat{\theta}_{t+1} \\
&\quad +\mu\beta(1-\rho)\left(\frac{\kappa}{q}\right)\left(\frac{1-bp}{1-b}\right)\hat{r}_t
\end{aligned}$$

Let

$$A = \mu\left(\frac{1}{1-b}\right)\left(\frac{\kappa}{q}\right).$$

Then

$$\begin{aligned}
\hat{\xi}_t &= A(1-\alpha)\hat{\theta}_t - A\beta(1-\rho)\mathrm{E}_t\left[1-\alpha-bp\right]\hat{\theta}_{t+1} \\
&\quad +A\beta(1-\rho)\left(1-b\theta q\right)\hat{r}_t \\
&\quad +A\left(\frac{b}{1-b}\right)\left[1-\beta(1-\rho)(1-p)\rho_b\right]\hat{b}_t
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mu}_t &= z_t - \hat{\xi}_t = z_t - A(1-\alpha)\hat{\theta}_t + A\beta(1-\rho)\mathrm{E}_t\left(1-\alpha-b\theta q\right)\hat{\theta}_{t+1} \\
&\quad -A\beta(1-\rho)\left(1-b\theta q\right)\hat{r}_t - B\hat{b}_t,
\end{aligned}$$

where

$$B \equiv A\left(\frac{b}{1-b}\right)\left[1-\beta(1-\rho)(1-p)\rho_b\right].$$

Evaluated around the efficient steady state,

$$A = \left(\frac{1}{\alpha}\right)\left(\frac{\kappa\bar{V}}{\rho\bar{N}}\right).$$

From (9) - (11) and the model of price adjustment,

$$\hat{\mu}_t = z_t - \mu\left(a_1\hat{\theta}_t - \beta a_2 E_t\hat{\theta}_{t+1} - \beta a_3\hat{r}_t + B\hat{b}_t\right) \tag{17}$$

$$\pi_t = \beta E_t\pi_{t+1} - \delta\hat{\mu}_t. \tag{18}$$

The Fisher equation is

$$i_t = \hat{r}_t - E_t\pi_{t+1}. \tag{19}$$

These eight equations, when combined with the production function and a specification of monetary policy, can be solved for $\hat{c}_t$, $\hat{y}_t$, $\hat{n}_t$, $\hat{u}_t$, $\hat{v}_t$, $\hat{\theta}_t$, $\hat{\mu}_t$, $\hat{r}_t$, π_t, and i_t. We proceed to obtain a version of the model that consists of three structural equations in π_t, $\hat{u}_t$, and $\hat{\theta}_t$.

Define $\eta \equiv (1-\rho)\left(\bar{N}/\bar{u}\right)$ and $\rho_u \equiv (1-\rho)\left(1-\rho\bar{N}/\bar{u}\right)$. We can use (14), (15) and (16) to write

$$\hat{\theta}_t = -\left(\frac{1}{\alpha\rho\eta}\right)\left(\hat{u}_{t+1} - \rho_u\hat{u}_t\right). \tag{20}$$

Since $\hat{v}_t = \hat{\theta}_t + \hat{u}_t$,

$$\hat{v}_t = -\left(\frac{1}{\alpha\rho\eta}\right)\hat{u}_{t+1} + \left[1 + \rho_u\left(\frac{1}{\alpha\rho\eta}\right)\right]\hat{u}_t. \tag{21}$$

The constant returns to scale technology for whoelsale good's production implies, when linearized, that $\hat{y}_t = \hat{n}_t + \hat{z}_t$. Thus, $\hat{y}_t$ can be eliminated from (12) to yield

$$\hat{c}_t = \left(\frac{Y}{C}\right)(1-w^u)\hat{n}_t + \left(\frac{Y}{C}\right)z_t - \left(\frac{\kappa V}{C}\right)\hat{v}_t. \tag{22}$$

We can now use (21) to write the goods market clearing condition (22) as

$$\begin{aligned}\hat{c}_t \;\; = \;\; & -\left(\frac{Y}{C}\right)(1-w^u)\left(\frac{1}{\eta}\right)\hat{u}_{t+1} + \left(\frac{Y}{C}\right)z_t + \left(\frac{\kappa V}{C}\right)\left(\frac{1}{\alpha\rho\eta}\right)\hat{u}_{t+1} \\ & -\left(\frac{\kappa V}{C}\right)\left[1 + \rho_u\left(\frac{1}{\alpha\rho\eta}\right)\right]\hat{u}_t,\end{aligned}$$

or

$$\hat{c}_t = \varphi_1\hat{u}_{t+1} - \varphi_2\hat{u}_t + \left(\frac{Y}{C}\right)z_t, \tag{23}$$

where, $\varphi_1 \equiv -\left(\bar{Y}/\eta\bar{C}\right)\left[1 - w^u - \left(\kappa\bar{V}/\alpha\rho\bar{Y}\right)\right] = -\left(\bar{Y}/\eta\bar{C}\right)\delta_1$ and $\varphi_2 \equiv \left(\kappa\bar{Y}/\alpha\rho\eta\bar{C}\right)(\alpha\rho\eta + \rho_u) = \left(\bar{Y}/\eta\bar{C}\right)\delta_2$.

Eliminating $\hat{c}_t$ from (13) yields

$$\begin{aligned}\hat{u}_{t+1} \;\; = \;\; & \gamma E_t\hat{u}_{t+2} + (1-\gamma)\hat{u}_t - \left[\frac{1}{\sigma(\varphi_1+\varphi_2)}\right](i_t - E_t\pi_{t+1}) \\ & + \left(\frac{1}{\varphi_1+\varphi_2}\right)\left(\frac{Y}{C}\right)(E_t z_{t+1} - \hat{z}_t),\end{aligned} \tag{24}$$

where $\gamma = \varphi_1/(\varphi_1+\varphi_2)$. It is straightforward to show that when the efficiency condition (2) holds, $\gamma = \beta/(1+\beta)$ so that (24) becomes

$$\begin{aligned}\hat{u}_{t+1} &= \left(\frac{\beta}{1+\beta}\right) E_t\hat{u}_{t+2} + \left(\frac{1}{1+\beta}\right)\hat{u}_t - \left(\frac{1}{1+\beta}\right)\left(\frac{\eta\bar{C}}{\delta_2\bar{N}}\right)\left(\frac{1}{\sigma}\right)(i_t - E_t\pi_{t+1}) \\ &+ \left(\frac{1}{1+\beta}\right)\left(\frac{\eta}{\delta_2}\right)(E_t\hat{z}_{t+1} - \hat{z}_t) \qquad (25)\end{aligned}$$

Using (17) to eliminate real marginal cost from the inflation equation (18),

$$\pi_t = \beta E_t\pi_{t+1} + \delta\left[a_1\hat{\theta}_t - \beta a_2 E_t\hat{\theta}_{t+1} + \beta a_3\left(i_t - E_t\pi_{t+1}\right) + B\hat{b}_t - \hat{z}_t\right]. \qquad (26)$$

Equations (20), (25), and (26) gives three equilibrium conditions in terms of unemployment, labor market tightness, and inflation that could be combined with a specification of monetary policy.

A version of the model that is even more similar to a standard new Keynesian specification can be obtained by using (20) to eliminate $\hat{\theta}_t$ from (26) to yield two structural equations, an unemployment based IS relationship and an unemployment based Phillips curve. These two relationships are

$$\hat{u}_{t+1} = \gamma E_t\hat{u}_{t+2} + (1-\gamma)\hat{u}_t - \left[\frac{1}{\sigma(\varphi_1+\varphi_2)}\right]\hat{r}_t - \left(\frac{1}{\varphi_1+\varphi_2}\right)\left(\frac{\bar{Y}}{\bar{C}}\right)(1-\rho_z)\hat{z}_t \qquad (27)$$

and

$$\begin{aligned}\pi_t &= \beta E_t\pi_{t+1} + \left(\frac{\delta\mu}{\alpha\rho\eta}\right)\left[a_2\beta\left(E_t\hat{u}_{t+2} - \rho_u\hat{u}_{t+1}\right) - a_1\left(\hat{u}_{t+1} - \rho_u\hat{u}_t\right)\right] \\ &+ \beta\delta\mu a_3\hat{r}_t + \delta B\hat{b}_t - \delta\hat{z}_t \qquad (28)\end{aligned}$$

where

$$\begin{aligned}\gamma &\equiv \varphi_1/\left(\varphi_1+\varphi_2\right) \\ \rho_u &\equiv (1-\rho)(1-\rho\bar{N}/\bar{u}) \\ \eta &\equiv (1-\rho)\left(\bar{N}/\bar{u}\right) \\ \varphi_1 &\equiv -\left(\bar{Y}/\eta\bar{C}\right)\left[1 - w^u - \left(\kappa\bar{V}/\alpha\rho\bar{Y}\right)\right] \\ \varphi_2 &\equiv \left(\kappa\bar{Y}/\alpha\rho\eta\bar{C}\right)\left(\alpha\rho\eta + \rho_u\right) \\ a_1 &= [(1-\alpha)/(1-b)]\left(\kappa\bar{V}/\rho\bar{N}\right) \\ a_2 &= a_1\left[(1-\rho)/(1-\alpha)\right](1-\alpha-\rho\bar{N}/\bar{u}) \\ a_3 &= a_1\left[(1-\rho)/(1-\alpha)\right](1-b\rho\bar{N}/\bar{u}) \\ B &= [b/(1-b)]\left[1-\mu w^u + \mu\beta(1-\rho)\left(\kappa\bar{V}/\bar{u}\right)\right].\end{aligned}$$

A primary difference from a standard Phillips curve is that inflation is affected by current and expected future values unemployment, as well as by the real rate of interest.[3]

4 Natural levels (flexible prices)

From the marginal cost expression when prices are flexible,

$$(1-b_t)\frac{Z_t}{\mu} = (1-b_t)\,w^u + \frac{\kappa}{\varphi}\theta_t^{1-\alpha} - (1-\rho)\left(\frac{1}{R_t}\right)E_t\left(\frac{\kappa}{\varphi}\theta_{t+1}^{1-\alpha} - \kappa b_t \theta_{t+1}\right). \tag{29}$$

Comparing this to (1) shows that the equilibrium will be efficient if $b = 1-\alpha$ (the Hosios condition), $\hat{b}_t = 0$, and $\mu = 1$. Linearizing (29) around the efficient steady-state, and letting a superscript e denote the efficient equilibrium, one obtains

$$\hat{z}_t = (1-\alpha)\frac{\kappa\bar{V}}{\alpha\rho\bar{N}}\left(\hat{\theta}_t^e - \beta\rho_u E_t\hat{\theta}_{t+1}^e\right) + \beta\delta_2\hat{r}_t^e. \tag{30}$$

Since $\delta_1 = -\beta\delta_2$, so we also can rewrite (30) as

$$\hat{z}_t = (1-\alpha)\left(\frac{\kappa\bar{V}}{\alpha\rho\bar{N}}\right)\left(\hat{\theta}_t^e - \beta\rho_u E_t\hat{\theta}_{t+1}^e\right) - \delta_1\hat{r}_t^e. \tag{31}$$

The IS relationship (25) holds whether prices are sticky or flexible, so

$$\begin{aligned}\hat{u}_{t+1}^e \quad = \quad & \left(\frac{\beta}{1+\beta}\right)E_t\hat{u}_{t+2}^e + \left(\frac{1}{1+\beta}\right)\hat{u}_t^e - \left(\frac{1}{1+\beta}\right)\left(\frac{\eta}{\delta_2}\right)\left(\frac{\bar{C}}{\bar{N}}\right)\left(\frac{1}{\sigma}\right)r_t^e \\ & + \left(\frac{1}{1+\beta}\right)\left(\frac{\eta}{\delta_2}\right)\left(E_t z_{t+1} - \hat{z}_t\right).\end{aligned}$$

It will be convenient to rewrite this in terms of employment as

$$\hat{z}_t = \delta_2\left[(1+\beta)\hat{n}_t^e - \beta E_t\hat{n}_{t+1}^e - \hat{n}_{t-1}^e\right] - \left(\frac{\bar{C}}{\bar{Y}}\right)\left(\frac{1}{\sigma}\right)\hat{r}_t^e + E_t\hat{z}_{t+1}. \tag{32}$$

5 Welfare

To derive an approximation to the representative agent's utility, it is necessary to first introduce some additional notation. For any variable X_t, let $\bar{X}$ be its steady-state value and let $\hat{X}_t = \log\left(X_t/\bar{X}\right)$ be the log deviation of X_t around its steady-state value. Using a second order Taylor approximation,

$$X_t - \bar{X} = \bar{X}\left(\frac{X_t}{\bar{X}} - 1\right) \approx \bar{X}\left(\hat{X}_t + \frac{1}{2}\hat{X}_t^2\right). \tag{33}$$

[3] To obtain a single equation representing the contraints on monetary policy (as long as interest rates do not appear in the central bank's objective function), the IS relationship can be used to eliminate the real interest rate from the Phillips curve. In this case, inflation will depend on current and expected future *changes* in the unemployment rate as well as its level.

Employing this notation, we develop a second order approximation to the utility of the representative household. In doing so, we found the work of Edge (2003) very helpful.

5.1 Household utility

The second order approximation to household utility, which is a function of total consumption, is

$$\begin{aligned} U(C_t) &\simeq U(\bar{C}) + U_c \bar{C}^m \left(\hat{c}_t^m + \frac{1}{2} (\hat{c}_t^m)^2 \right) - w^u U_c \bar{N} \left(\hat{n}_t + \frac{1}{2} \hat{n}_t^2 \right) \\ &\quad + \frac{1}{2} U_{cc} \left(\bar{C}^m \right)^2 \left[\hat{c}_t^m - \left(\frac{w^u \bar{N}}{\bar{C}^m} \right) \hat{n}_t \right]^2 . \end{aligned} \tag{34}$$

If we define

$$\hat{\sigma} = -\frac{U_{cc} \bar{C}}{U_c} \left(\frac{\bar{C}^m}{\bar{C}} \right) = \sigma \left(\frac{\bar{C}^m}{\bar{C}} \right)$$

Then (34) becomes

$$\begin{aligned} U(C_t) &\simeq U(\bar{C}) + U_c \bar{C}^m \left(\hat{c}_t^m + \frac{1}{2} (\hat{c}_t^m)^2 \right) - U_c w^u \bar{N} \left(\hat{n}_t + \frac{1}{2} \hat{n}_t^2 \right) \\ &\quad - \frac{1}{2} \hat{\sigma} U_c \bar{C}^m \left[\hat{c}_t^m - \left(\frac{w^u \bar{N}}{\bar{C}^m} \right) \hat{n}_t \right]^2 \end{aligned} \tag{35}$$

5.1.1 Second order expansion for market consumption

Market clearing requires

$$C_t^m D_t = Z_t N_t - \kappa V_t,$$

where

$$D_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} di$$

measures the dispersion of relative prices. So

$$\begin{aligned} \frac{C_t^m D_t}{\bar{C}^m} &= \frac{Z_t N_t}{\bar{C}^m} - \frac{\kappa V_t}{\bar{C}^m} \\ &\approx 1 + \frac{\bar{N}}{\bar{C}^m} \left[(\hat{z}_t + \hat{n}_t) + \frac{1}{2} (\hat{z}_t + \hat{n}_t)^2 \right] - \frac{\kappa \bar{V}}{\bar{C}^m} \left(\hat{v}_t + \frac{1}{2} \hat{v}_t^2 \right). \end{aligned} \tag{36}$$

5.1.2 Second order expansion for vacancies

Since $V_t = \theta_t U_t$,

$$\hat{v}_t = \hat{\theta}_t + \hat{u}_t$$

is exact, but to express this in terms of $\hat{n}_t$, we need to use the correct second order approximation for unemployment in terms of employment. Since $U_t = 1-(1-\rho)N_{t-1}$,

$$\hat{u}_t + \frac{1}{2}\hat{u}_t^2 = -(1-\rho)\left(\frac{\bar{N}}{\bar{U}}\right)\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right) = -\eta\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right). \tag{37}$$

This then implies $\hat{u}_t^2 = \eta^2 \hat{n}_{t-1}^2$, so (37) becomes

$$\hat{u}_t = -\eta\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right) - \frac{1}{2}\hat{u}_{t-1}^2 = -\eta\hat{n}_{t-1} - \frac{1}{2}\eta\left(1+\eta\right)\hat{n}_{t-1}^2,$$

and

$$\hat{v}_t = \hat{\theta}_t + \hat{u}_t = \hat{\theta}_t - \eta\hat{n}_{t-1} - \frac{1}{2}\eta\left(1+\eta\right)\hat{n}_{t-1}^2,$$

while $\hat{v}_t^2 = \left(\hat{\theta}_t - \eta\hat{n}_{t-1}\right)^2$. Hence,

$$\begin{aligned}
\hat{v}_t + \frac{1}{2}\hat{v}_t^2 &= \hat{\theta}_t - \eta\hat{n}_{t-1} - \frac{1}{2}\eta\left(1+\eta\right)\hat{n}_{t-1}^2 + \frac{1}{2}\left(\hat{\theta}_t - \eta\hat{n}_{t-1}\right)^2 \\
&= \left(\hat{\theta}_t + \frac{1}{2}\hat{\theta}_t^2\right) - \eta\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right) - \eta\hat{\theta}_t\hat{n}_{t-1}.
\end{aligned} \tag{38}$$

5.1.3 Second order approximation for labor market tightness

Now we need a second order approximation for labor market tightness, $\hat{\theta}_t$. From $N_t = (1-\rho)N_{t-1} + \varphi\theta_t^{\alpha}U_t$, we obtain

$$\begin{aligned}
\bar{N}\left(1 + \hat{n}_t + \frac{1}{2}\hat{n}_t^2\right) &= (1-\rho)\bar{N}\left(1 + \hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right) \\
&\quad + \varphi\bar{\theta}^{\alpha}\bar{U}\left(1 + \alpha\hat{\theta}_t + \frac{1}{2}\alpha^2\hat{\theta}_t^2\right)\left(1 - \eta\hat{n}_{t-1} - \frac{1}{2}\eta\hat{n}_{t-1}^2\right).
\end{aligned}$$

This implies

$$\left(\hat{n}_t + \frac{1}{2}\hat{n}_t^2\right) = (1-\rho)\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right) + \rho\left(\alpha\hat{\theta}_t - \eta\hat{n}_{t-1} + \frac{1}{2}\alpha^2\hat{\theta}_t^2 - \alpha\eta\hat{\theta}_t\hat{n}_{t-1} - \frac{1}{2}\eta\hat{n}_{t-1}^2\right)$$

which, since $(1-\rho) - \rho\eta = (1-\rho)\left(1 - \rho\bar{N}/\bar{U}\right) = \rho_u$, can be written as

$$\alpha\rho\left(\hat{\theta}_t + \frac{1}{2}\alpha\hat{\theta}_t^2\right) = \left(\hat{n}_t - \rho_u\hat{n}_{t-1}\right) + \frac{1}{2}\left(\hat{n}_t^2 - \rho_u\hat{n}_{t-1}^2\right) + \alpha\rho\eta\hat{\theta}_t\hat{n}_{t-1}. \tag{39}$$

But (39) implies that

$$\hat{\theta}_t\hat{n}_{t-1} = \left(\frac{1}{\alpha\rho}\right)\left(\hat{n}_t - \rho_u\hat{n}_{t-1}\right)\hat{n}_{t-1}, \tag{40}$$

and

$$\hat{\theta}_t^2 = \left(\frac{1}{\alpha\rho}\right)^2\left(\hat{n}_t - \rho_u\hat{n}_{t-1}\right)^2. \tag{41}$$

Now, using (39)-(41) in (38),

$$\hat{v}_t + \frac{1}{2}\hat{v}_t^2 = \left(\frac{1}{\alpha\rho}\right)(\hat{n}_t - \rho_u \hat{n}_{t-1}) + \frac{1}{2}\left(\frac{1}{\alpha\rho}\right)(\hat{n}_t^2 - \rho_u \hat{n}_{t-1}^2) - \eta\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right) + \frac{1}{2}(1-\alpha)\left(\frac{1}{\alpha\rho}\right)^2 (\hat{n}_t - \rho_u \hat{n}_{t-1})^2 \tag{42}$$

5.1.4 The approximation for market consumption

We can now use the expressing for vacancies given by (42) in (36) to obtain

$$\begin{aligned} \frac{C_t^m D_t}{\bar{C}^m} \approx\ & 1 + \frac{\bar{N}}{\bar{C}^m}\left[(\hat{z}_t + \hat{n}_t) + \frac{1}{2}(\hat{z}_t + \hat{n}_t)^2\right] \\ & - \frac{\kappa \bar{V}}{\bar{C}^m}\left[\left(\frac{1}{\alpha\rho}\right)(\hat{n}_t - \rho_u \hat{n}_{t-1}) + \frac{1}{2}\left(\frac{1}{\alpha\rho}\right)(\hat{n}_t^2 - \rho_u \hat{n}_{t-1}^2)\right] \\ & - \frac{\kappa \bar{V}}{\bar{C}^m}\left[\frac{1}{2}(1-\alpha)\left(\frac{1}{\alpha\rho}\right)^2 (\hat{n}_t - \rho_u \hat{n}_{t-1})^2 - \eta\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right)\right]. \end{aligned}$$

Using the approximation $\ln(1+x) \approx x - \frac{1}{2}x^2$,

$$\begin{aligned} \ln\left(\frac{C_t^m D_t}{\bar{C}^m}\right) \equiv\ & \hat{c}_t^m + \hat{d}_t = \left(\frac{\bar{N}}{\bar{C}^m}\right)\left[(\hat{z}_t + \hat{n}_t) - \left(\frac{\kappa \bar{V}}{\alpha\rho\bar{N}}\right)(\hat{n}_t - \rho_u \hat{n}_{t-1})\right] \\ & + \frac{1}{2}\left(\frac{\bar{N}}{\bar{C}^m}\right)(\hat{z}_t + \hat{n}_t)^2 - \frac{1}{2}\frac{\kappa \bar{V}}{\bar{C}^m}\left(\frac{1}{\alpha\rho}\right)(\hat{n}_t^2 - \rho_u \hat{n}_{t-1}^2) \\ & - \frac{\kappa \bar{V}}{\bar{C}^m}\left[\frac{1}{2}(1-\alpha)\left(\frac{1}{\alpha\rho}\right)^2 (\hat{n}_t - \rho_u \hat{n}_{t-1})^2 - \eta\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right)\right] \\ & - \frac{1}{2}\left(\frac{\bar{N}}{\bar{C}^m}\right)^2\left[(\hat{z}_t + \hat{n}_t) - \left(\frac{\kappa \bar{V}}{\alpha\rho\bar{N}}\right)(\hat{n}_t - \rho_u \hat{n}_{t-1})\right]^2. \end{aligned}$$

Hence

$$\begin{aligned} \hat{c}_t^m =\ & -\hat{d}_t + \left(\frac{\bar{N}}{\bar{C}^m}\right)\left[(\hat{z}_t + \hat{n}_t) - \left(\frac{\kappa \bar{V}}{\alpha\rho\bar{N}}\right)(\hat{n}_t - \rho_u \hat{n}_{t-1})\right] \\ & + \frac{1}{2}\left(\frac{\bar{N}}{\bar{C}^m}\right)(\hat{z}_t + \hat{n}_t)^2 - \frac{1}{2}\frac{\kappa \bar{V}}{\bar{C}^m}\left(\frac{1}{\alpha\rho}\right)(\hat{n}_t^2 - \rho_u \hat{n}_{t-1}^2) \\ & - \frac{\kappa \bar{V}}{\bar{C}^m}\left[\frac{1}{2}(1-\alpha)\left(\frac{1}{\alpha\rho}\right)^2 (\hat{n}_t - \rho_u \hat{n}_{t-1})^2 - \eta\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right)\right] \\ & - \frac{1}{2}\left(\frac{\bar{N}}{\bar{C}^m}\right)^2\left[(\hat{z}_t + \hat{n}_t) - \left(\frac{\kappa \bar{V}}{\alpha\rho\bar{N}}\right)(\hat{n}_t - \rho_u \hat{n}_{t-1})\right]^2. \end{aligned} \tag{43}$$

and

$$(\hat{c}_t^m)^2 = \left(\frac{\bar{N}}{\bar{C}^m}\right)^2\left[(\hat{z}_t + \hat{n}_t) - \left(\frac{\kappa \bar{V}}{\alpha\rho\bar{N}}\right)(\hat{n}_t - \rho_u \hat{n}_{t-1})\right]^2. \tag{44}$$

since $\hat{d}_t$ is already second order,

5.2 Evaluating utility

Repeating (35),

$$
\begin{aligned}
U(C_t) &= U(\bar{C}) + U_c\bar{C}^m \left(\hat{c}_t^m + \frac{1}{2}\left(\hat{c}_t^m\right)^2 \right) - U_c w^u \bar{N} \left(\hat{n}_t + \frac{1}{2}\hat{n}_t^2 \right) \\
&\quad -\frac{1}{2}\hat{\sigma} U_c \bar{C}^m \left[\hat{c}_t^m - \left(\frac{w^u \bar{N}}{\bar{C}^m} \right) \hat{n}_t \right]^2 .
\end{aligned}
$$

Recalling that

$$
\delta_1 = 1 - w^u - \frac{\kappa\bar{V}}{\alpha\rho\bar{N}}
$$

and noting that

$$
\frac{\kappa\bar{V}}{\alpha\rho\bar{N}}\left(\rho_u + \alpha\rho\eta\right) = (1-\rho)\frac{\kappa\bar{V}}{\alpha\bar{U}}\left(\alpha - 1 + \frac{\bar{U}}{\rho\bar{N}} \right) = \delta_2,
$$

we can express utility, after some simplification, as

$$
\begin{aligned}
U(C_t) &= U(\bar{C}) - U_c\bar{C}^m\hat{d}_t + U_c\bar{C}^m\left(\frac{\bar{N}}{\bar{C}^m}\right)\left(\hat{z}_t + \delta_1\hat{n}_t + \delta_2\hat{n}_{t-1}\right) \\
&\quad +U_c\bar{C}^m\left(\frac{\bar{N}}{\bar{C}^m}\right)\left(\frac{1}{2}\hat{z}_t^2 + \hat{z}_t\hat{n}_t + \frac{1}{2}\delta_1\hat{n}_t^2 + \frac{1}{2}\delta_2\hat{n}_{t-1}^2\right) \\
&\quad -\frac{1}{2}U_c\bar{C}^m(1-\alpha)\frac{\kappa\bar{V}}{\bar{C}^m}\left[\left(\frac{1}{\alpha\rho}\right)^2\left(\hat{n}_t - \rho_u\hat{n}_{t-1}\right)^2\right] \\
&\quad -\frac{1}{2}\hat{\sigma}U_c\bar{C}^m\left(\frac{\bar{N}}{\bar{C}^m}\right)^2\left(\hat{z}_t + \delta_1\hat{n}_t + \delta_2\hat{n}_{t-1}\right)^2 .
\end{aligned}
\tag{45}
$$

5.3 The present discounted value of utility

From (45),

$$
\begin{aligned}
\sum_{i=0}^{\infty}\beta^i U(C_{t+i}) &= \frac{U(\bar{C})}{1-\beta} - U_c\bar{C}^m\sum_{i=0}^{\infty}\beta^i\hat{d}_{t+i} + U_c\bar{N}\sum_{i=0}^{\infty}\beta^i\left(\hat{z}_{t+i} + \delta_1\hat{n}_{t+i} + \delta_2\hat{n}_{t+i-1}\right) \\
&\quad +\frac{1}{2}U_c\bar{N}\sum_{i=0}^{\infty}\beta^i\left(\hat{z}_{t+i}^2 + \delta_1\hat{n}_{t+i}^2 + \delta_2\hat{n}_{t+i-1}^2\right) + U_c\bar{N}\sum_{i=0}^{\infty}\beta^i\hat{z}_{t+i}\hat{n}_{t+i} \\
&\quad -\frac{1}{2}U_c(1-\alpha)\kappa\bar{V}\left(\frac{1}{\alpha\rho}\right)^2\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i} - \rho_u\hat{n}_{t+i-1}\right)^2 \\
&\quad -\frac{1}{2}\hat{\sigma}U_c\bar{C}^m\left(\frac{\bar{N}}{\bar{C}^m}\right)^2\sum_{i=0}^{\infty}\beta^i\left(\hat{z}_{t+i} + \delta_1\hat{n}_{t+i} + \delta_2\hat{n}_{t+i-1}\right)^2 .
\end{aligned}
$$

5.3.1 First order terms

In section 1, it is shown that in the efficient equilibrium, $\delta_1 = -\beta\delta_2$. Therefore, the first order terms become

$$\begin{aligned} & -U_c\bar{N}\delta_2 \sum_{i=0}^{\infty} \beta^i \left(\beta\hat{n}_{t+i} - \hat{n}_{t+i-1}\right) + t.i.p. \\ = & -U_c\bar{N}\delta_2 \left\{\beta\hat{n}_t - \hat{n}_{t-1} + \beta^2\hat{n}_{t+1} - \beta\hat{n}_t + ...\right\} \\ = & \; U_c\bar{N}\delta_2\hat{n}_{t-1} \end{aligned}$$

which is independent of policy.

5.3.2 Second order terms

The second order terms are

$$\begin{aligned} X_t \equiv & -U_c\bar{C}^m \sum_{i=0}^{\infty} \beta^i \hat{d}_{t+i} + \frac{1}{2}U_c\bar{N} \sum_{i=0}^{\infty} \beta^i \left(\delta_1\hat{n}_{t+i}^2 + \delta_2\hat{n}_{t+i-1}^2\right) \\ & +U_c\bar{N} \sum_{i=0}^{\infty} \beta^i \hat{z}_{t+i}\hat{n}_{t+i} - \frac{1}{2}U_c(1-\alpha)\kappa\bar{V}\left(\frac{1}{\alpha\rho}\right)^2 \sum_{i=0}^{\infty} \beta^i \left(\hat{n}_{t+i} - \rho_u\hat{n}_{t+i-1}\right)^2 \\ & -\frac{1}{2}\hat{\sigma}U_c\bar{C}^m \left(\frac{\bar{N}}{\bar{C}^m}\right)^2 \sum_{i=0}^{\infty} \beta^i \left(\hat{z}_{t+i} + \delta_1\hat{n}_{t+i} + \delta_2\hat{n}_{t+i-1}\right)^2, \end{aligned}$$

plus a term in $\hat{z}_t^2$ that is independent of policy.

Price dispersion term Up to second order,

$$\sum_{i=0}^{\infty} \beta^i d_{t+i} = -\frac{\varepsilon}{2\delta} \sum_{i=0}^{\infty} \beta^i \pi_{t+i}^2$$

where δ is the standard new Keynesian coefficient giving the elasticity of inflation with respect to marginal cost.

Term in employment squared The next term is

$$\begin{aligned} \frac{1}{2}U_c\bar{N} \sum_{i=0}^{\infty} \beta^i \left(\delta_1\hat{n}_{t+i}^2 + \delta_2\hat{n}_{t+i-1}^2\right) &= -\frac{1}{2}\delta_2 U_c\bar{N} \sum_{i=0}^{\infty} \beta^i \left(\beta\hat{n}_{t+i}^2 - \hat{n}_{t+i-1}^2\right) \\ &= -\frac{1}{2}\delta_2 U_c\bar{N}\hat{n}_{t-1}^2 \end{aligned}$$

which is independent of policy.

Cross products with productivity This leaves us to deal with

$$X_t \equiv U_c\bar{N}\sum_{i=0}^{\infty}\beta^i\hat{z}_{t+i}\hat{n}_{t+i} - \frac{1}{2}U_c\bar{N}(1-\alpha)\left(\frac{\kappa\bar{V}}{\bar{N}}\right)\left(\frac{1}{\alpha\rho}\right)^2\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i}-\rho_u\hat{n}_{t+i-1}\right)^2$$
$$-\frac{1}{2}\hat{\sigma}U_c\bar{C}^m\left(\frac{\bar{N}}{\bar{C}^m}\right)^2\sum_{i=0}^{\infty}\beta^i\left(\hat{z}_{t+i}-\beta\delta_2\hat{n}_{t+i}+\delta_2\hat{n}_{t+i-1}\right)^2,$$

or

$$X_t \equiv U_{cc}\bar{N}\sum_{i=0}^{\infty}\beta^i\hat{z}_{t+i}\hat{n}_{t+i} - \frac{1}{2}U_c\bar{N}(1-\alpha)\left(\frac{\kappa\bar{V}}{\bar{N}}\right)\left(\frac{1}{\alpha\rho}\right)^2\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i}-\rho_u\hat{n}_{t+i-1}\right)^2$$
$$-\hat{\sigma}\delta_2U_c\bar{C}^m\left(\frac{\bar{N}}{\bar{C}^m}\right)^2\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i-1}-\beta\hat{n}_{t+i}\right)\hat{z}_{t+i}$$
$$-\frac{1}{2}\hat{\sigma}\delta_2U_c\bar{C}^m\left(\frac{\bar{N}}{\bar{C}^m}\right)^2\sum_{i=0}^{\infty}\beta^i\left(\beta\hat{n}_{t+i}-\hat{n}_{t+i-1}\right)^2. \quad (46)$$

Start with the third term involving the cross product expression, $(\hat{n}_{t-1}-\beta\hat{n}_t)\hat{z}_t$. Using (32),

$$(\hat{n}_{t-1}-\beta\hat{n}_t)\hat{z}_t = \hat{n}_{t-1}\hat{z}_t-\beta\hat{n}_t\hat{z}_t=\hat{n}_{t-1}\hat{z}_t$$
$$-\beta\hat{n}_t\left\{\delta_2\left[(1+\beta)\hat{n}_t^e-\beta E_t\hat{n}_{t+1}^e-\hat{n}_{t-1}^{e*}\right]-\left(\frac{\bar{C}}{\bar{\bar{Y}}}\right)\left(\frac{1}{\sigma}\right)\hat{r}_t^e\right\}$$
$$-\beta\hat{n}_tE_t\hat{z}_{t+1}$$

so

$$\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i-1}-\beta\hat{n}_{t+i}\right)\hat{z}_{t+i} = \hat{n}_{t-1}\hat{z}_t-\beta\hat{n}_tE_t\hat{z}_{t+1}$$
$$-\beta\hat{n}_t\left\{\delta_2\left[(1+\beta)\hat{n}_t^e-\beta E_t\hat{n}_{t+1}^e-\hat{n}_{t-1}^{e*}\right]-\left(\frac{\bar{C}}{\bar{\bar{Y}}}\right)\left(\frac{1}{\sigma}\right)\hat{r}_t^e\right\}$$
$$+\beta\hat{n}_tE_t\hat{z}_{t+1}-\beta^2E_t\hat{n}_{t+1}\hat{z}_{t+2}$$
$$-\beta^2E_t\hat{n}_{t+1}\left\{\delta_2\left[(1+\beta)\hat{n}_{t+1}^e-\beta E_t\hat{n}_{t+2}^e-\hat{n}_t^{e*}\right]-\left(\frac{\bar{C}}{\bar{\bar{Y}}}\right)\left(\frac{1}{\sigma}\right)\hat{r}_{t+1}^e\right\}$$
$$+....$$
$$= \hat{n}_{t-1}\hat{z}_t-\sum_{i=0}^{\infty}\beta^i\left\{\delta_2\left[(1+\beta)\hat{n}_{t+i}^e-\beta E_t\hat{n}_{t+i+1}^e-\hat{n}_{t+i-1}^{e*}\right]-\left(\frac{\bar{C}}{\bar{\bar{Y}}}\right)\left(\frac{1}{\sigma}\right)\hat{r}_{t+i}^e\right\}\hat{n}_{t+i}$$

But $\hat{n}_{t-1}\hat{z}_t$ is independent of policy at time t.

So the terms in X_t involving cross product terms with the shocks (the first and third terms in

46) become, in expected present value terms,[4]

$$
\begin{aligned}
& U_c\bar{N}\left[\sum_{i=0}^{\infty}\beta^i\hat{n}_{t+i}\hat{z}_{t+i}-\hat{\sigma}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i-1}-\beta\hat{n}_{t+i}\right)\hat{z}_{t+i}\right] \\
= \; & U_c\bar{N}\sum_{i=0}^{\infty}\beta^i\left\{\hat{n}_{t+i}\hat{z}_{t+i}+\beta\hat{\sigma}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2^2\left[(1+\beta)\hat{n}^e_{t+i}-\beta E_t\hat{n}^e_{t+i+1}-\hat{n}^{e*}_{t+i-1}\right]\hat{n}_{t+i}-\beta\delta_2 r^e_{t+i}\hat{n}_{t+i}\right\}.
\end{aligned}
$$

>From (31),

$$
\hat{n}_t\hat{z}_t=\left[(1-\alpha)\left(\frac{\kappa\bar{V}}{\alpha\rho\bar{N}}\right)\left(\hat{\theta}^e_t-\beta\rho_u E_t\hat{\theta}^e_{t+1}\right)-\delta_1\hat{r}^e_t\right]\hat{n}_t.
$$

Using this to replace $\hat{n}_t\hat{z}_t$ in the expression for the present discounted value expression yields

$$
\begin{aligned}
& U_c\bar{N}\sum_{i=0}^{\infty}\beta^i\left[(1-\alpha)\left(\frac{\kappa\bar{V}}{\alpha\rho\bar{N}}\right)\left(\hat{\theta}^e_{t+i}-\beta\rho_u E_t\hat{\theta}^e_{t+i+1}\right)-\delta_1\hat{r}^e_{t+i}\right]\hat{n}_{t+i} \\
& +U_c\bar{N}\sum_{i=0}^{\infty}\beta^i\left\{\beta\hat{\sigma}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2^2\left[(1+\beta)\hat{n}^e_{t+i}-\beta E_t\hat{n}^e_{t+i+1}-\hat{n}^{e*}_{t+i-1}\right]\hat{n}_{t+i}-\beta\delta_2\hat{r}^e_{t+i}\hat{n}_{t+i}\right\}.
\end{aligned}
$$

The real interest rate terms cancel out since $\delta_1-\beta\delta_2=0$, so we are left with

$$
\begin{aligned}
& U_c\bar{N}\sum_{i=0}^{\infty}\beta^i\left[(1-\alpha)\left(\frac{\kappa\bar{V}}{\alpha\rho\bar{N}}\right)\left(\hat{\theta}^e_{t+i}-\beta\rho_u E_t\hat{\theta}^e_{t+i+1}\right)\right]\hat{n}_{t+i} \\
& +U_c\bar{N}\hat{\sigma}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2^2\sum_{i=0}^{\infty}\beta^{i+1}\left[(1+\beta)\hat{n}^e_{t+i}-\beta E_t\hat{n}^e_{t+i+1}-\hat{n}^{e*}_{t+i-1}\right]\hat{n}_{t+i}
\end{aligned}
$$

for the cross product terms involving $\hat{n}$ and $\hat{z}$. But

$$
\left(\hat{\theta}^e_t-\beta\rho_u E_t\hat{\theta}^e_{t+1}\right)\hat{n}_t=\hat{\theta}^e_t\left(\rho_u\hat{n}_{t-1}+\alpha\rho\hat{\theta}_t\right)-\beta E_t\hat{\theta}^e_{t+1}\rho_u\hat{n}_t, \tag{47}
$$

so when we take present discounted values, the $\hat{\theta}^e_{t+i}\rho_u\hat{n}_{t+i-1}$ and $\beta E_t\hat{\theta}^e_{t+i+1}\rho_u\hat{n}_{t+i}$ terms will cancel (except for the first, $\rho_u\hat{\theta}^e_t\hat{n}_{t-1}$, but that is independent of policy. Thus,

$$
\sum_{i=0}^{\infty}\beta^i\left(\hat{\theta}^e_{t+i}-\beta\rho_u E_t\hat{\theta}^e_{t+i+1}\right)\hat{n}_{t+i}=\alpha\rho\sum_{i=0}^{\infty}\beta^i\hat{\theta}_{t+i}\hat{\theta}^e_{t+i}+\rho_u\hat{n}_{t-1}\hat{\theta}^e_t.
$$

[4]The coefficient on $\hat{r}^e_t\hat{n}_t$ is $\beta\hat{\sigma}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2\left(\frac{\bar{C}}{\bar{Y}}\right)\left(\frac{1}{\sigma}\right)$, but

$$
\begin{aligned}
\beta\hat{\sigma}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2\left(\frac{\bar{C}}{\bar{Y}}\right)\left(\frac{1}{\sigma}\right) &= \beta\sigma\left(\frac{C^m}{\bar{C}}\right)\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2\left(\frac{\bar{C}}{\bar{Y}}\right)\left(\frac{1}{\sigma}\right) \\
&= \beta\delta_2.
\end{aligned}
$$

Hence, for the PDV of the cross product terms we have

$$\begin{aligned}\sum_{i=0}^{\infty}\beta^i B_{t+i} &= U_c\bar{N}(1-\alpha)\left(\frac{\kappa\bar{V}}{\bar{N}}\right)\sum_{i=0}^{\infty}\beta^i\hat{\theta}_{t+i}\hat{\theta}^e_{t+i} \\ &+U_c\bar{N}\hat{\sigma}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2\sum_{i=0}^{\infty}\beta^{i+1}\left[(1+\beta)\hat{n}^e_{t+i}-\beta E_t\hat{n}^e_{t+i+1}-\hat{n}^{e*}_{t+i-1}\right]\hat{n}_{t+i} \qquad (48)\end{aligned}$$

Cross products with other variables Define

$$\begin{aligned}\sum_{i=0}^{\infty}\beta^i G_{t+i} &\equiv -\frac{1}{2}U_c\bar{N}(1-\alpha)\left(\frac{\kappa\bar{V}}{\bar{N}}\right)\left(\frac{1}{\alpha\rho}\right)^2\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i}-\rho_u\hat{n}_{t+i-1}\right)^2 \\ &-\frac{1}{2}\hat{\sigma}U_c\bar{C}^m\left(\frac{\bar{N}}{\bar{C}^m}\right)^2\sum_{i=0}^{\infty}\beta^i\left(\delta_1\hat{n}_{t+i}+\delta_2\hat{n}_{t+i-1}\right)^2 \qquad (49)\end{aligned}$$

as the present discounted value of the cross product terms among the variables in X_t (i.e., terms 2 and 4 of 46). From (41), this can be written as

$$\begin{aligned}\sum_{i=0}^{\infty}\beta^i G_{t+i} &\equiv -\frac{1}{2}U_c\bar{N}(1-\alpha)\left(\frac{\kappa\bar{V}}{\bar{N}}\right)\sum_{i=0}^{\infty}\beta^i\hat{\theta}^2_{t+i} \\ &-\frac{1}{2}\hat{\sigma}U_c\bar{N}\left(\frac{\bar{N}}{\bar{C}^m}\right)\sum_{i=0}^{\infty}\beta^i\left(\delta_1\hat{n}_{t+i}+\delta_2\hat{n}_{t+i-1}\right)^2\end{aligned}$$

For future reference, notice that

$$\sum_{i=0}^{\infty}\beta^i\left(\delta_1\hat{n}_{t+i}+\delta_2\hat{n}_{t+i-1}\right)^2=\delta_2^2\sum_{i=0}^{\infty}\beta^i\left(-\beta\hat{n}_{t+i}+\hat{n}_{t+i-1}\right)^2=\delta_2^2\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i-1}-\beta\hat{n}_{t+i}\right)^2.$$

5.3.3 Collecting results on all second order product terms

Adding together the PDVs of B_t and G_t,

$$\begin{aligned}X_t &= (1-\alpha)U_c\bar{N}\left(\frac{\kappa\bar{V}}{\bar{N}}\right)\sum_{i=0}^{\infty}\beta^i\hat{\theta}_{t+i}\hat{\theta}^e_{t+i} \\ &+\hat{\sigma}U_c\bar{N}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2^2\sum_{t=0}^{\infty}\beta^{i+1}\left[(1+\beta)\hat{n}^e_{t+i}-\beta E_t\hat{n}^e_{t+i+1}-\hat{n}^{e*}_{t+i-1}\right]\hat{n}_{t+i} \\ &-\frac{1}{2}U_c\bar{N}(1-\alpha)\left(\frac{\kappa\bar{V}}{\bar{N}}\right)\sum_{i=0}^{\infty}\beta^i\hat{\theta}^2_{t+i} \\ &-\frac{1}{2}\hat{\sigma}U_c\bar{N}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2^2\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i-1}-\beta\hat{n}_{t+i}\right)^2+t.i.p \qquad (50)\end{aligned}$$

or

$$
\begin{aligned}
X_t &= -\frac{1}{2}(1-\alpha)U_c\left(\frac{\kappa\bar{V}}{\bar{N}}\right)\sum_{i=0}^{\infty}\beta^i\left(\hat{\theta}_{t+i}-\hat{\theta}^e_{t+i}\right)^2 \\
&+\hat{\sigma}U_c\bar{N}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2^2\sum_{t=0}^{\infty}\beta^{i+1}\left[(1+\beta)\hat{n}^e_{t+i}-\beta E_t\hat{n}^e_{t+i+1}-\hat{n}^{e*}_{t+i-1}\right]\hat{n}_{t+i} \\
&-\frac{1}{2}\hat{\sigma}U_c\bar{N}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2^2\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i-1}-\beta\hat{n}_{t+i}\right)^2+t.i.p \qquad (51)
\end{aligned}
$$

Terms involving the CRRA Start by focusing on all the terms that are multiplied by $\hat{\sigma}$. These are $\hat{\sigma}U_c\bar{N}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2^2$ times

$$
J_t \equiv \sum_{i=0}^{\infty}\beta^{i+1}\left[(1+\beta)\hat{n}^e_{t+i}-\beta E_t\hat{n}^e_{t+i+1}-\hat{n}^{e*}_{t+i-1}\right]\hat{n}_{t+i}-\frac{1}{2}\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i-1}-\beta\hat{n}_{t+i}\right)^2. \qquad (52)
$$

First, note that

$$
\left[(1+\beta)\hat{n}^e_t-\beta E_t\hat{n}^e_{t+1}-\hat{n}^{e*}_{t-1}\right]\hat{n}_t=\left[\left(\beta\hat{n}^e_t-\hat{n}^{e*}_{t-1}\right)-\left(\beta E_t\hat{n}^e_{t+1}-\hat{n}^e_t\right)\right]\hat{n}_t.
$$

It follows that

$$
\begin{aligned}
\sum_{i=0}^{\infty}\beta^{i+1}\left[(1+\beta)\hat{n}^e_{t+i}-\beta E_t\hat{n}^e_{t+i+1}-\hat{n}^{e*}_{t+i-1}\right]\hat{n}_{t+i} &= \sum_{i=0}^{\infty}\beta^{i+1}\left[\left(\beta\hat{n}^e_{t+i}-\hat{n}^e_{t+i-1}\right)-\left(\beta E_t\hat{n}^e_{t+i+1}-\hat{n}^e_{t+i}\right)\right]n_{t+i} \\
&= \beta\left[\left(\beta\hat{n}^e_t-\hat{n}^{e*}_{t-1}\right)-\left(\beta E_t\hat{n}^e_{t+1}-\hat{n}^e_t\right)\right]n_t \\
&\quad +\beta^2\left[\left(\beta\hat{n}^e_{t+1}-\hat{n}^{e*}_t\right)-\left(\beta E_t\hat{n}^e_{t+2}-\hat{n}^e_{t+1}\right)\right]n_{t+1}+... \\
&= \left(\beta\hat{n}^e_t-\hat{n}^{e*}_{t-1}\right)\hat{n}_{t-1}-\left(\beta\hat{n}^e_t-\hat{n}^{e*}_{t-1}\right)\left(\hat{n}_{t-1}-\beta n_t\right) \\
&\quad -\beta\left(\beta E_t\hat{n}^e_{t+1}-\hat{n}^e_t\right)\left(\hat{n}_t-\beta n_{t+1}\right)+...+t.i.p. \\
&= \left(\beta\hat{n}^e_t-\hat{n}^{e*}_{t-1}\right)\hat{n}_{t-1} \\
&\quad -\sum_{i=0}^{\infty}\beta^i\left(\beta\hat{n}^e_{t+i}-\hat{n}^e_{t+i-1}\right)\left(\hat{n}_{t+i-1}-\beta n_{t+i}\right)
\end{aligned}
$$

Notice first term $\left(\beta\hat{n}^e_t-\hat{n}^{e*}_{t-1}\right)\hat{n}_{t-1}$ is independent of policy.

Using these in (52) yields

$$
\begin{aligned}
J_t &= -\sum_{i=0}^{\infty}\beta^i\left(\beta\hat{n}^e_{t+i}-\hat{n}^{e*}_{t+i-1}\right)\left(\hat{n}_{t+i-1}-\beta n_{t+i}\right)-\frac{1}{2}\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i-1}-\beta\hat{n}_{t+i}\right)^2+t.i.p. \\
&= \sum_{i=0}^{\infty}\beta^i\left(\hat{n}^e_{t+i-1}-\beta\hat{n}^e_{t+i}\right)\left(\hat{n}_{t+i-1}-\beta n_{t+i}\right)-\frac{1}{2}\sum_{i=0}^{\infty}\beta^i\left(\hat{n}_{t+i-1}-\beta\hat{n}_{t+i}\right)^2+t.i.p. \\
&= -\frac{1}{2}\sum_{i=0}^{\infty}\beta^i\left[\left(\hat{n}_{t+i-1}-\beta\hat{n}_{t+i}\right)-\left(\hat{n}^e_{t+i-1}-\beta\hat{n}^e_{t+i}\right)\right]^2+t.i.p..
\end{aligned}
$$

So the terms involving $\hat{\sigma}$ are equal to

$$-\frac{1}{2}\hat{\sigma}U_c\bar{N}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2^2\sum_{i=0}^{\infty}\beta^i\left[\left(\hat{n}_{t+i-1}-\hat{n}^e_{t+i-1}\right)-\beta\left(\hat{n}_{t+i}-\hat{n}^e_{t+i}\right)\right]^2. \tag{53}$$

5.4 Final results

Collecting results, we can now write (51) as

$$\begin{aligned} X_t &= -(1-\alpha)\frac{1}{2}U_c\bar{N}\left(\frac{\kappa\bar{V}}{\bar{N}}\right)\sum_{i=0}^{\infty}\beta^i\left(\hat{\theta}_{t+i}-\hat{\theta}^e_{t+i}\right)^2 \\ &\quad -\frac{1}{2}\hat{\sigma}U_c\bar{N}\left(\frac{\bar{N}}{\bar{C}^m}\right)\delta_2^2\sum_{i=0}^{\infty}\beta^i\left[\left(\hat{n}_{t+i-1}-\hat{n}^e_{t+i-1}\right)-\beta\left(\hat{n}_{t+i}-\hat{n}^e_{t+i}\right)\right]^2+t.i.p \end{aligned}$$

Recalling that $\hat{\sigma}=\sigma\bar{C}^m/\bar{C}$, the correct second order approximation for welfare is therefore given by

$$\begin{aligned} \sum_{i=0}^{\infty}\beta^iU(C_t) &= \frac{U(\bar{C})}{1-\beta}-\frac{\varepsilon}{2\delta}U_c\bar{C}\sum_{i=0}^{\infty}\beta^i\pi_t^2 \\ &\quad -(1-\alpha)\frac{1}{2}U_c\bar{N}\left(\frac{\kappa\bar{V}}{\bar{N}}\right)\sum_{i=0}^{\infty}\beta^i\left(\hat{\theta}_{t+i}-\hat{\theta}^e_{t+i}\right)^2 \\ &\quad -\frac{1}{2}\sigma U_c\bar{N}\left(\frac{\bar{N}}{\bar{C}}\right)\delta_2^2\sum_{i=0}^{\infty}\beta^i\left[\left(\hat{n}_{t+i-1}-\hat{n}^e_{t+i-1}\right)-\beta\left(\hat{n}_{t+i}-\hat{n}^e_{t+i}\right)\right]^2+t.i.p. \end{aligned}$$

It is useful at this point to recall that (23) implies we can write, in gap terms,

$$\tilde{c}_t=-\left(\frac{\bar{N}}{\eta\bar{C}}\right)\left(\delta_1\tilde{u}_{t+1}+\delta_2\tilde{u}_t\right)=\left(\frac{\bar{N}}{\bar{C}}\right)\left(\delta_1\tilde{n}_t+\delta_2\tilde{n}_{t-1}\right).$$

Using the efficient condition, this becomes

$$\tilde{c}_t=-\left(\frac{\bar{N}}{\bar{C}}\right)\delta_2\left(\beta\tilde{n}_t-\tilde{n}_{t-1}\right).$$

Hence, the last term in the welfare approximation is equal to

$$-\frac{1}{2}\sigma U_c\bar{C}\sum_{i=0}^{\infty}\beta^i\tilde{c}_t^2.$$

Thus, we have that

$$\begin{aligned} \sum_{i=0}^{\infty}\beta^iU(C_t) &= \frac{U(\bar{C})}{1-\beta}-\frac{\varepsilon}{2\delta}U_c\bar{C}\sum_{i=0}^{\infty}\beta^i\left[\pi_t^2+(1-\alpha)\left(\frac{\delta}{\varepsilon}\right)\left(\frac{\kappa\bar{V}}{\bar{C}}\right)\tilde{\theta}_{t+i}^2+\sigma\left(\frac{\delta}{\varepsilon}\right)\tilde{c}_{t+i}^2\right] \\ &\quad +t.i.p. \end{aligned}$$

which is produces (23) of the text.

Since our structural model is expressed in terms of inflation and unemployment, it will be useful to similarly express the loss function in terms of these two variables. Since $\hat{u}_{t+1} = -\eta \hat{n}_t$, and $\hat{\theta}_t = \left(\frac{1}{\alpha\rho}\right)(\hat{n}_t - \rho_u \hat{n}_{t-1})$. In addition,

$$\tilde{c}_t = -\left(\frac{\bar{N}}{\bar{C}}\right)\delta_2\left(\beta\tilde{n}_t - \tilde{n}_{t-1}\right) = \left(\frac{\delta_2\bar{N}}{\eta\bar{C}}\right)\left(\beta\tilde{u}_{t+1} - \tilde{u}_t\right),$$

so we can write the loss solely in terms of unemployment relative to the efficient level $\hat{u}^e$. Doing so yields (26) of the text.