Web Appendix to "Learning about Risk and Return: A Simple Model of Bubbles and Crashes" by William A. Branch and George W. Evans

Proof of Proposition 1. We use the stochastic approximation results described in Evans and Honkapohja (1998, 2001) and (Marcet and Sargent 1989*b*). Set $\gamma_{1,t} = \gamma_{2,t} = t^{-1}$ and define $z_t = p_t - \theta'_{t-1}X_{t-1} + u_t = \left(T(\theta_{t-1};\sigma^2_{t-1}) - \theta_{t-1}\right)X_{t-1} - a\beta\sigma^2_{t-1}v_t + u_t$. Then (12)-(15) in the main text, for the case of exogenous supply, can be re-written as the equations (20)-(22) in the print Appendix to the paper, reproduced here:

$$\begin{aligned}
\theta_t &= \theta_{t-1} + t^{-1}S^{-1}_{t-1}X_{t-1}\left(X'_{t-1}(T(\theta_{t-1};\sigma^2_{t-1}) - \theta_{t-1})' - a\beta\sigma^2_{t-1}v_t\right)\\
S_t &= S_{t-1} + t^{-1}\left(\frac{t}{t+1}\left(X_tX'_t - S_{t-1}\right)\right)\\
\sigma^2_t &= \sigma^2_{t-1} + t^{-1}\left(z_tz'_t - \sigma^2_{t-1}\right).
\end{aligned}$$

Defining $\phi_t = (\theta_t, \mathrm{vec}(S_t), \sigma^2_t)'$, and then using the framework of (Evans and Honkapohja 2001), it is straightforward to verify that the ODE (ordinary differential equation) associated with the asymptotic behavior of the learning algorithm (16), i.e.

$$\frac{d\phi}{d\tau} = h(\phi).$$

is given by

$$\begin{aligned}
h_\theta &= S^{-1}M(\theta,\sigma^2)(T(\theta;\sigma^2) - \theta)'\\
h_S &= M(\theta,\sigma^2) - S\\
h_{\sigma^2} &= (T(\theta;\sigma^2) - \theta)M(\theta,\sigma^2)(T(\theta;\sigma^2) - \theta)' + \sigma^2_u + (a\beta\sigma^2)^2\sigma^2_v - \sigma^2,
\end{aligned}$$

and where $M(\theta,\sigma^2) = EX_t(\theta,\sigma^2)X_t(\theta,\sigma^2)'$. Locally stable REE under (12)-(15) are associated with stable rest points of the ODE. The Jacobian matrix of this ODE, evaluated at the REE, provides the relevant stability conditions:

$$\begin{pmatrix}
\beta(1+c)-1 & \beta k & 0 & 0 & 0 & 0 & -\beta a s_0\\
0 & 2\beta c - 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0\\
\frac{\partial M(1,2)}{\partial k} & \frac{\partial M(1,2)}{\partial c} & 0 & -1 & 0 & 0 & 0\\
\frac{\partial M(1,2)}{\partial k} & \frac{\partial M(1,2)}{\partial c} & 0 & 0 & -1 & 0 & 0\\
\frac{\partial M(2,2)}{\partial k} & \frac{\partial M(2,2)}{\partial c} & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 2a^2\beta^2\sigma^2_v\sigma^2 - 1
\end{pmatrix}.$$

Local stability requires all eigenvalues to have negative real parts. The Jacobian matrix has eigenvalues $-1+2c\beta, -1+\beta+c\beta, -1+2a^2\beta^2\sigma^2_v\sigma^2$, and repeated values of -1. The root $-1+2a^2\beta^2\sigma^2_v\sigma^2$ corresponds to the derivative of the quadratic $\sigma^2_u + \sigma^2_v(a\beta\sigma^2)^2 - \sigma^2$ and it is easily verified that this is negative at the lower root $\sigma^2 = \sigma^2_L$. At the fundamentals solution $c = 0$, the other nonzero roots are -1 and $-1+\beta$. Since $0 < \beta < 1$ all roots of the Jacobian matrix are negative, which implies E-stability (and thus stability under learning). At the RE bubble solution $c = \beta^{-1}$, there is one root equal to one, which implies E-instability.

Proof of Proposition 2 We proceed by first noting that under constant-gain learning $\gamma_{1,t} = \gamma_1 > 0$, $\gamma_{2,t} = \gamma_2 > 0$, it is possible to rewrite the real-time learning algorithms (12)-(14) in the form

$$\phi_t^\gamma = \phi_{t-1}^\gamma + \gamma \mathcal{H}(\phi_{t-1}^\gamma, \bar{X}_t)$$

where $\bar{X}_t' = (1, p_t, p_{t-1}, u_t, v_t)'$. The components of $\mathcal{H}$ are implicitly defined by (12)-(14), with a fixed multiplicative term γ_2/γ_1 incorporated into (14). The superscript γ has been added to the parameter estimates ϕ^γ to emphasize their dependence on the gain $\gamma = \gamma_1$. In order to make a comparison between the solutions to the continuous time ODE and the discrete time recursive algorithm, we need to define a corresponding continuous time sequence for ϕ_t^γ, denoted $\phi^\gamma(\tau)$, given by $\phi^\gamma(\tau) = \phi_t^\gamma$ if $\tau_t^\gamma \leq \tau < \tau_{t+1}^\gamma$, where $\tau_t^\gamma = t\gamma$.

We sketch the proof to this proposition by making use of Proposition 7.8 of Evans and Honkapohja, itself a re-statement of Benveniste, Metivier, and Priouret (1990, Theorem 7, Chp. 4.4.3, Part II). The proposition in the text is based on the proposition stated below. Let D be an open set containing the fundamentals REE parameters $\theta^*, S^*, \sigma^{2*}$. In the case of exogenous share supply, the actual law of motion followed by price is

$$p_t = T(k_{t-1}, c_{t-1}; \sigma_{t-1}^2)X_{t-1} - \beta a \sigma_{t-1}^2 v_t.$$

It is clearly the case that the state dynamics are conditionally linear and can be written as

$$\bar{X}_t \equiv \begin{bmatrix} X_t \\ X_{t-1} \\ u_t \\ v_t \end{bmatrix} = \begin{bmatrix} A(\phi_{t-1}) & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \bar{X}_{t-1} + \begin{bmatrix} B(\phi_{t-1}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} W_t$$

where $I, 0$ are conformable matrices, and

$$X_t = A(\phi_{t-1})X_{t-1} + B(\phi_{t-1})W_t$$

with $X_t' = (1, p_t)', W_t' = (1, u_t, v_t)'$. The validity of the proposition depends on the following properties as established in (Evans and Honkapohja 2001).

P1 W_t is iid with finite absolute moments.

P2 For any compact $Q \subset D$, $\sup_{\phi \in Q} |B(\phi)| \leq M$ and $\sup_{\phi \in Q} |A(\phi)| \leq \rho < 1$, and $|\cdot|$ is an appropriately defined matrix norm.

P3 For any compact $Q \subset D$, $\exists C, q$ s.t. $\forall \phi \in Q$ and for all t $|\mathcal{H}(\phi, x)| \leq C(1 + |x|^q)$.

P4 For any compact $Q \subset D$, $\mathcal{H}(\phi, x)$ is twice continuously differentiable with bounded second derivatives.

P5 $h(\phi)$ has continuous first and second derivatives on D.

Here $h(\phi)$ is as defined earlier except that the σ^2 component of $h(\phi)$ is multiplied by the fixed ratio γ_2/γ_1. The conditional linearity simplifies verification of these conditions. Proposition 7.5 of (Evans and Honkapohja 2001) shows that conditions M1-M5 of their Proposition 7.8 are implied by P1-P2. For their assumption A3' we also make use of the remark on p. 155, which shows that P4 is sufficient.

For given ϕ let $p_t(\phi) = T(k, c; \sigma^2)X_{t-1} - \beta a\sigma^2 v_t$ and let $X_t(\phi)' = (1, p_t(\phi))'$. Then $X_t(\phi)$ is stationary for ϕ sufficiently close to the fundamentals REE. Therefore, fix D to be an open set around $(\theta^*, S^*, \sigma^{2*})$ such that $\forall(\theta, S, \sigma^2) \in D$, we have: (1) $(\theta^*, S^*, \sigma^{2*})$ are such that σ^{2*} is the unique solution in D to the quadratic $\sigma_u^2 + (a\beta\sigma^2)^2 - \sigma^2 = 0$, θ^* is the unique fixed point of $T(\theta; \sigma^2)$ on D with $\sigma^2 = \sigma^{2*}$, $S^* = EX_t(\phi^*)X_t(\phi^*)'$, (2) for some $\tilde{\varepsilon} > 0$, $\det(S) \geq \tilde{\varepsilon} > 0$, and (3) $k(1+c) \geq -y_0$ and $-1 < c < \bar{c} < \beta^{-1/2}$.

Write $\bar{X}_t = \bar{A}(\phi_{t-1})\bar{X}_{t-1} + \bar{B}(\phi_{t-1})W_t$, where $\bar{A}, \bar{B}$ are given above. Clearly the eigenvalues of $\bar{A}$ consist of zero and the eigenvalues of A. The set D is defined so that the roots of $A(\phi)$ are inside the unit circle implying $\bar{A}(\phi)$ will also have roots with modulus less than one. It is straightforward to verify that assumptions P1-P5 hold. We use the following result from (Evans and Honkapohja 2001):

PROPOSITION 3: *[EH(2001), Proposition 7.8] Assume P1-P5. Consider the normalized random variables* $U^\gamma(\tau) = \gamma^{-1/2}\left[\phi^\gamma(\tau) - \tilde{\phi}(\tau, \phi_0)\right]$. *As* $\gamma \to 0$, *the process* $U^\gamma(\tau)$, $0 \leq \tau \leq T$, *converges weakly to the solution* $U(\tau)$ *of the stochastic differential equation*

$$dU(\tau) = D_\phi h(\tilde{\phi}(\tau, \phi_0))U(\tau)d\tau + \mathcal{R}^{1/2}(\tilde{\phi}(\tau, \phi_0))dW(\tau)$$

with initial condition $U(0) = 0$, *where* $W(\tau)$ *is a standard vector Wiener process, and* $\mathcal{R}$ *is a covariance matrix whose* i, j*th elements are*

$$\mathcal{R}^{ij}(\phi) = \sum_{k=-\infty}^{\infty} Cov\left[\mathcal{H}^i(\phi, \bar{X}_k^\phi), \mathcal{H}^j(\phi, \bar{X}_0^\phi)\right]$$

Finally Proposition 2 can be established by noting that the solution to the stochastic differential equation $U(\tau)$ has the following properties

$$\begin{aligned} EU(\tau) &= 0 \\ \frac{dVar(U(\tau))}{d\tau} &= D_\phi h(\tilde{\phi}(\tau, \phi_0))V_u(\tau) + V_u D_\phi h(\tilde{\phi}(\tau, \phi_0))' + \mathcal{R}(\tilde{\phi}(\tau, \phi_0)), \end{aligned}$$

where $V_u = Var(U(\tau))$.

Details on Approximating the Mean Dynamics With Endogenous Share Supply. Under learning we continue to have

$$p_t = \beta\left(y_0 + k_{t-1}(1 + c_{t-1})\right) + \beta c_{t-1}^2 p_{t-1} - \beta a\sigma_{t-1}^2 z_{st},$$

but when share supply may become endogenous additional care is required to construct the mean dynamics. The condition for exogenous supply, $s_0 \leq \Phi p_t$, is satisfied if and only if

$$s_0 \leq \Phi \frac{\beta\left(k(1+c)+y_0\right)}{1+\beta a\sigma^2\Phi(1+v_t)} + \Phi \frac{\beta c^2}{1+\beta a\sigma^2\Phi(1+v_t)} p_{t-1}, \text{ or}$$

(23) $$s_0\Phi^{-1} + s_0\beta a\sigma^2(1+v_t) \leq \beta\left(k(1+c)+y_0\right) + \beta c^2 p_{t-1}.$$

Given $\tilde{\theta} = (k, c; \sigma^2)$, equations (5), (6) and (23) specify $p_t = F(p_{t-1}, v_t; \tilde{\theta})$. For computing mean dynamics the complication is that whether (23) is satisfied, and thus whether (5) or (6) applies, depends on v_t.

Mean dynamics are computed by fixing $\tilde{\theta}$ and $\tilde{S}$ and computing the ODE, where the expectation is taken over v_t and $p_t(\tilde{\theta})$, the p_t process for fixed $\tilde{\theta}$. In general this must be done using the process given by (5), (6) and (23), and for any given $\tilde{\theta}$ one must take account of the possibility that either regime will occur, depending on v_t. However, at least for "small" v_t, a reasonable approximation would be to split the $\tilde{\theta}$ space into two regions: in one region the probability is high that (for the given $\tilde{\theta}$) the $p_t(\tilde{\theta})$ process will be given by (5), and in the other region the probability is high that the $p_t(\tilde{\theta})$ process will be given by (6).

For the region defuned by equation (5), $p_t(\tilde{\theta})$ converges to a stationary AR(1) with mean

$$Ep_t(\tilde{\theta}) = \frac{\beta\left(k(1+c)+y_0 - a\sigma^2 s_0\right)}{1-\beta c^2} \equiv \bar{p}_H,$$

provided $\beta c^2 < 1$. If $\beta c^2 > 1$ the condition $s_0 \leq \Phi p_t$ is satisfied (for $\lim_{t\to\infty} Ep_t(\tilde{\theta})$). For $\beta c^2 < 1$ the condition is satisfied, using the above expression for $Ep_t(\tilde{\theta})$ provided

$$s_0\Phi^{-1} + s_0\beta a\sigma^2 \leq \beta\left(k(1+c)+y_0\right) + \beta c^2 \bar{p}_H.$$

Here we have set $v_t = 0$, and replaced p_{t-1} by its mean under (5). The condition can be rewritten as

$$\sigma^2 \leq \bar{\sigma}_H^2(c,k), \text{ where}$$
$$\bar{\sigma}_H^2(c,k) = (s_0\beta a)^{-1}\left\{\beta\left(k(1+c)+y_0\right) - s_0\Phi^{-1} + \beta c^2\bar{p}_H\right\}.$$

For the region defined by equation (6) the linear approximation of the $p_t(\tilde{\theta})$ process is of the form

(24) $$p_t = \frac{\beta\left(k(1+c)+y_0\right)}{1+\beta a\sigma^2\Phi} + \frac{\beta c^2}{1+\beta a\sigma^2\Phi} p_{t-1} - \delta v_t,$$

which has mean

$$Ep_t = \bar{p}_L \equiv \frac{\beta\left(k(1+c)+y_0\right)}{1-\beta c^2+\beta a\sigma^2\Phi}.$$

Here

$$\delta = \frac{\beta^2 a \sigma^2 \Phi \left(k(1+c) + y_0 + \beta c^2 \bar{p}_L\right)}{\left(1 + \beta a \sigma^2 \Phi\right)^2}$$

Based on this mean, the condition $s_0 > \Phi p_t$ for (6) (with approximation (24)) will be satisfied when

$$\sigma^2 > \bar{\sigma}_L^2(c,k), \text{ where}$$
$$\bar{\sigma}_L^2(c,k) = (s_0 \beta a)^{-1} \left\{\beta \left(k(1+c) + y_0\right) - s_0 \Phi^{-1} + \beta c^2 \bar{p}_L\right\},$$

where we again set $v_t = 0$ and where we set p_{t-1} at its mean under (24). Since $\bar{p}_L < \bar{p}_H$ we have $\bar{\sigma}_L^2(c,k) < \bar{\sigma}_H^2(c,k)$. Thus when $\sigma^2 > \bar{\sigma}_H^2(c,k)$ and the distribution of v_t has small enough support, it is very likely that the (approximate) dynamics (24) will be followed.

In the main text we present numerical results for the mean dynamics based on the above approximation. Thus, for $\sigma^2 \leq \bar{\sigma}_H^2(c,k)$, we assume the mean dynamics are based on exogenous supply. For $\sigma^2 > \bar{\sigma}_H^2(c,k)$ the mean dynamics are instead assumed to be given by the alternative mean dynamics based on (24). Note for (24) the corresponding mapping from perceived law of motion to the actual law of motion has k, c components

$$(k,c) \to \left(\frac{\beta\left(k(1+c) + y_0\right)}{1 + \beta a \sigma^2 \Phi}, \frac{\beta c^2}{1 + \beta a \sigma^2 \Phi}\right).$$

and there is a corresponding expression for the σ^2 component of the ODE:

$$h_{\sigma^2} = \left(T(\theta;\sigma^2) - \theta\right) M(\theta, S, \sigma^2) \left(T(\theta;\sigma^2) - \theta\right)' + \sigma_u^2 + \delta^2 \sigma_v^2$$

It is worth remarking that this procedure ignores the chance that the process will have endogenous supply when $\sigma^2 \leq \bar{\sigma}_H^2(c,k)$ and it ignores the chance that it will have exogenous supply when $\sigma^2 > \bar{\sigma}_H^2(c,k)$. Within and near the region $\bar{\sigma}_L^2(c,k) < \sigma^2 < \bar{\sigma}_H^2(c,k)$ the approximation will be at its worst, since both regimes will have a significant chance of arising. But in order to provide intuition for the real time learning results, this approximation suffices.

Procedure for Computing the Confidence Ellipses. Here we outline the procedure. Details on the general procedure are given in Evans and Honkapohja (2001, Chp. 14, p. 348-356). The confidence ellipsoids assume that the parameter estimates k_t, c_t will be distributed asymptotically normal. Under similar assumptions to those for Proposition 2 this property can be established formally.

In (Evans and Honkapohja 2001) it is shown that $\theta_t \sim N(\theta^*, \gamma V)$ for small γ and large t, where $\theta' = (k,c)'$ and V solves the matrix Riccati equation

$$D_\theta h(\bar{\phi})V + V(D_\theta h(\bar{\phi}))' = -\mathcal{R}_\theta(\bar{\phi})$$

where $\mathcal{R} = E\mathcal{H}(\phi)\mathcal{H}(\phi)'$ is as given in the proof to Proposition 2. Notice that the way this Riccati equation is expressed omits the $D_S h(\bar{\phi})$ and $D_{\sigma^2} h(\bar{\phi})$ terms. This is because R is a block diagonal matrix:

$$\mathcal{R} = E\mathcal{H}(\bar{\phi})\mathcal{H}(\bar{\phi})' = \begin{bmatrix} (a\beta)^2(\bar{\sigma}^2)^2\sigma_v^2 M^{-1} & 0 & 0 \\ 0 & Evec\mathcal{H}_R vec\mathcal{H}_R' & 0 \\ 0 & 0 & \sigma_u^2 + (a\beta)^2(\bar{\sigma}^2)^2\sigma_v^2 - \bar{\sigma}^2 \end{bmatrix}$$

where $M = EX_{t-1}X_{t-1}'$. The text solves V numerically, sets $\gamma_2/\gamma_1 = 2$, and plots the 50% and 95% concentration ellipses.

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