Housing Bubbles –Supplemental Appendices

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This set of appendices accompanies the paper 'Housing Bubbles'. In this supplementary material, we describe how modifying the model to allow for growth in the life-cycle income profile, an elastic supply of houses, and a rental market do not significantly alter, but rather enrich the main results of the paper.

Appendix II. Endowment Growth and Asymmetric Preferences

In the extension of this appendix we add two new parameters to the baseline model used in the main text. First, let ϕ denote a positive endowment of consumption goods that households receive at the beginning of the second period of life. Second, let ω denote the relative weight of housing services in the utility function. Formally, introducing these two parameters will imply that the budget constraint in the main text (3T) now becomes:[1]

$$c_{t+1}^{m} + a_{t+1} \leq P_{t+1} h_{t+1}^{y} - R_t d_t + \phi,$$

while the utility function now takes the form

$$U_t = \log\left(c_t^{y}\right) + \beta[\omega \log(h_{t+1}^{y}) + \log\left(c_{t+1}^{m}\right)] + \beta^2 \log\left(c_{t+2}^{o}\right).$$

Thus, this model nests the baseline under $\phi = 0$, and $\omega = 1$.

We introduce these innovations with a double objective. First, we show that the lower limit $\underline{\theta}$ is negatively related to ϕ. Hence, under more plausible income-profiles than the one assumed in the baseline model, stationary bubbles are possible for values of θ lower than the one implied by main text expression (36T). In second, we show that the size of the θ-region of coexistence of steady states with and without bubbles (i.e. $\overline{\theta} - \underline{\theta}$) depends positively on the relative weight of housing services, ω, in the utility function. That is, multiplicity of steady states is more plausible if housing services are more heavily valued by the households.

To simplify the algebra and to make the analysis less cumbersome, we first assume that $\omega = 1$ and look at the link between ϕ and $\underline{\theta}$. Then we set $\phi = 0$ and analyze how different values of ω affect the coexistence region $\overline{\theta} - \underline{\theta}$. Since some parts of the model's solution now become analytically far more complex we illustrate the results using some numerical simulations.

The effects of ϕ on $\underline{\theta}$. Assuming that $\phi > 0$ and solving for the high-valuation steady state equilibrium allocations, we find the following expression for the consumption of a constrained

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[1] We will use a 'T' after the number in reference to expressions contained in the main text.

young household

$$c^y = \frac{1+(1+\beta)(1+\beta+\psi) - \sqrt{[1+(1+\beta)(1+\beta+\psi)]^2 - 4(1+\psi)(1+\beta)^2}}{2(1+\beta)^2}, \tag{1}$$

where $\psi \equiv \frac{\theta\phi}{1-(1-\theta)R}$. We can then express the demand and supply of loans when young and middle-aged as follows

$$a = \frac{\beta}{1+\beta}\left[1 + \frac{1-c^y}{\psi}\right]\phi, \tag{2}$$

and

$$d = \frac{1-\theta}{\theta}\left(1-c^y\right). \tag{3}$$

Using the aggregate counterparts of (2) and (3) and imposing the loans market-clearing condition evaluated at $R = 1$, we find the new expression for the lower bound

$$\underline{\theta} = \frac{1+\beta}{1+2\beta+\frac{\beta\phi}{1-c^y}}. \tag{4}$$

As c^y in (1) is a positive function of ϕ,[2] we learn from (4) that $\frac{d\underline{\theta}}{d\phi} < 0$. Figure A1 displays the negative relationship between ϕ and $\underline{\theta}$. The intuition behind this is as follows. On the one hand, an upward sloping income profile raises consumption in the first period at the cost of reducing the volume of the young's cohort own funds devoted to house purchases. Since the borrowing constraint is binding in this regime, the latter implies that the young's demand for loans falls as ϕ rises (see (3)). On the other hand, a higher ϕ implies that the schedule for savings of the middle-aged shifts upwards due to a positive income effect. Formally, after imposing $R = 1$,we can write (2), as follows:

$$a = \frac{\beta}{1+\beta}\left[1-c^y-\phi\right].$$

Since the marginal propensity to consume future endowment is less than unity, we find that $\frac{da}{d\phi} > 0$.

To summarize, we find that allowing for an upward sloping life cycle income profile tends to raise the excess supply of loans at $R = 1$, which allows for the emergence of a bubble for lower values of θ. In other words, a higher ϕ tends to magnify the asset scarcity problem, widening the θ-region for which stationary bubbles can be sustained.

The effects of ω on $\overline{\theta} - \underline{\theta}$. For a given value of ω, the total expenditure on housing services

[2]Notice that $\psi = \phi$ if $R = 1$, so c^y can then be expressed as a function of β and ϕ only.

Figure A1. Endowment growth and Bubbles

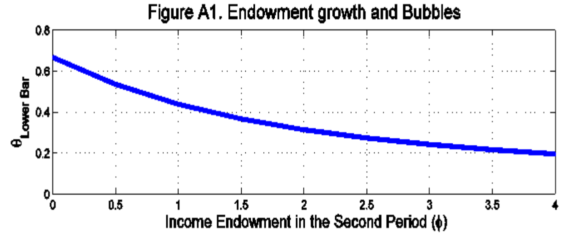
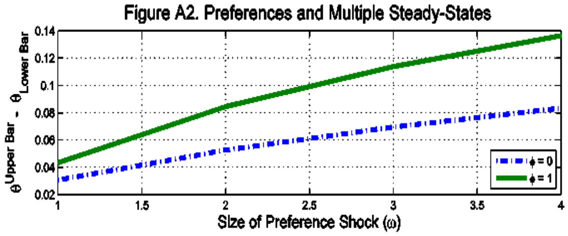


Figure A2. Preferences and Multiple Steady-States



by the young is given by

$$PH^y = \gamma' \beta \frac{1 + \omega + \beta}{\theta}, \tag{5}$$

where $\gamma' \equiv \left[1 + (1 + \omega)\beta + \beta^2\right]$. From expression (5) it follows that $\frac{dPh^y}{d\omega} > 0$. Then, combining the main text expressions of the binding borrowing constraint (5T) with (3T) and (8T) and imposing $R = 1$, we can write the aggregate supply of loans as

$$A = \frac{\beta}{1 + \beta} \theta PH^y.$$

Notice that the demand for loans is given by $D = (1 - \theta) PH^y$. Then, the equilibrium condition $A = B$ yields

$$\underline{\theta} = \frac{1 + \beta}{2 + \beta}.$$

Thus, the lower bound $\underline{\theta}$ is unaffected by ω. The intuition is simple. Since expression (5T) in the main text is binding and $R = 1$, a given change in the preferences parameter ω affects A and B in the same direction and magnitude, for both functions are proportional to the size of the down payment, θPH^y. Thus, both the excess supply of loans and the threshold $\underline{\theta}$ remain invariant to ω.

From the previous result it follows that the positive relationship between ω and $\overline{\theta} - \underline{\theta}$ is due to a positive effect of ω on $\overline{\theta}$. The aggregate demand and supply for loans by the young and the middle-aged generation, in a regime in which (5T) is not binding are given, respectively, by

$$D = \gamma' \beta \left[\frac{1}{R - 1}\omega - (1 + \beta)\right], \quad \text{and} \quad A = \gamma' \beta^2 R.$$

After some simple manipulations it follows that $\frac{\partial D}{\partial \omega} > 0$ and $\frac{\partial A}{\partial \omega} < 0$. These results are intuitive. First, for a given R, as housing services gain weight in the utility function the young increase their expenditure on housing and therefore their demand for loans. Second, the latter positive effect of ω on D implies that net wealth of the middle-aged falls as ω rises, thus reducing the supply of funding. As a result, we find that in equilibrium $\frac{dR}{d\omega} > 0$.

Next, as in the baseline model, we compute $\underline{\theta}$ by solving the equality $d = \left(1 - \overline{\theta}\right) Ph^y$ and imposing $\mu = 0$. This yields the following expression for $\underline{\theta}$,

$$\overline{\theta} = \frac{1 + \beta + \omega}{\omega} \frac{R - 1}{R}.$$

This expression helps explain the mechanism behind the positive effect of ω on $\overline{\theta}$, implicit in the numerical examples displayed in Figure A2. On the one hand, an increase in ω reduces $\overline{\theta}$ by augmenting the leverage ratio, d/Ph^y. (Recall that $\overline{\theta}$ is defined as the unique solution of the equality $d = \left(1 - \overline{\theta}\right) Ph^y$). That is, as ω increases the subsequent rise in the housing expenditure, Ph^y, is not compensated by an equal reduction in consumption, c^y, as it would be the case when the individual's borrowing is constrained. A rise in the leverage ratio implies that the borrowing limit is reached at a lower θ. On the other hand, an increase in ω raises R, which pushes down the leverage ratio. As shown in the simulations in Figure A2, this latter effect dominates, and a rise in ω reduces d/Ph^y, which means that the borrowing limit will only bind for a higher θ.

Appendix III. An Elastic Supply of Houses

Here we analyze the robustness of some of the central results to the assumption of a fully inelastic supply of houses. To this aim, we consider a very simple class of housing supply functions along which the total stock of houses at t can be expressed as a function of the house price at t. That is, we abstract from potential complexities arising from sluggish adjustment of the housing stock, dependence from past prices, etc. Also, for simplicity, we leave the construction sector unmodelled and just focus on the effects of a positive elasticity on the key equations determining the existence and dynamics of housing bubbles.

The following proposition shows that allowing for a positive elasticity in the supply of houses does not change *per se* the central features of the model.

Proposition A-III. Assume that the total stock of houses, H_t, is a function of the contemporaneous housing price, with $\frac{dH_t}{dP_t} > 0$ and with finite elasticity $\epsilon_t = \frac{dH_t}{dP_t}\frac{P_t}{H_t}$. Then, i) stationary bubbles exist if and only if $\theta > \frac{1+\beta}{1+2\beta}$, i.e. the same condition as when $\epsilon_t = 0$; ii) for any ϵ_t, the model exhibits oscillatory convergence around a steady state with a housing bubble, i.e. the model retains the same (qualitative) dynamic properties as when $\epsilon_t = 0$.

Proof. i) We notice that neither the supply, equation (15T), nor the demand for loans, equation (13T), depend upon the particular process followed by the supply of houses, i.e. the value of ϵ_t. Thus, according to Propositions 1 and 2, $\theta > \frac{1+\beta}{1+2\beta}$ is still the necessary and sufficient condition for stationary pure and housing bubbles.

ii) In order to derive the analogous equation to (31T), we combine the housing market-clearing condition, $P_t H_t = \frac{1-\gamma}{\theta} + B_t^{HB}$, with expression (30T), which remains unaffected by ϵ_t, for the same reason emphasized in point i) above, to get

$$P_t H_t = (1-\gamma)\left(1 + \frac{\beta}{1+\beta}\frac{P_t}{P_{t-1}}\right), \tag{6}$$

from which we learn that,

$$\frac{dP_t}{dP_{t-1}} = -\frac{(1-\gamma)\frac{\beta}{1+\beta}\frac{P_t}{P_{t-1}^2}}{(1+\epsilon_t)H_t - (1-\gamma)\frac{\beta}{1+\beta}\frac{1}{P_{t-1}}}.$$

Using this last expression it is straightforward to verify that $\frac{dP_t}{dP_{t-1}} < 0$, and $\lim_{P_{t-1}\to P^{HB}} \left|\frac{dP_t}{dP_{t-1}}\right| < 1$ for any finite value of ϵ_t. Thus, within a housing bubble regime, the system converges with damped oscillations towards the steady state, exactly as in the benchmark case with $\epsilon_t = 0$.

These results are intuitive. Bubbles in our model arise as the result of the gap between the supply and the demand for funds. Such a gap is then devoted to sustain speculative investments in an equilibrium with bubbles. Crucially, the funds gap need not be affected in a substantial way by the particular process the supply of bubbly-assets, as follows in the example presented above. In fact, the elasticity of the supply of houses does not affect the funds gap, $A - D$, in the long run at all, while in the short run it affects it indirectly, via its effects on the growth rate of the housing price, but in a way that preserves its general dynamic properties. In this sense, we say that allowing for an elastic supply of houses does not affect *per se* any of the central results of the paper.

Appendix IV. A Rental Market

We here consider a simple extension of the model presented in Section I of the paper that incorporates a rental market. To introduce a demand for rental services, we modify the utility function of the model in the main text, (1T), as follows

$$U_t = \log\left(c_t^y\right) + \beta[\log(h_{t+1}^y + h_{t+1}^r) + \log\left(c_{t+1}^m\right)] + \beta^2 \log\left(c_{t+2}^o\right), \tag{7}$$

where h_{t+1}^r stands for the flow of housing services that a tenant obtains from a rented dwelling of size h_{t+1}^r. (Notice that housing services obtained from one's own stock of housing delivers the same utility as services obtained from rented houses.) Assuming that a tenant belonging to the young generation at t pays the rental price at time t, denoted by P_t^r, the flow of funds constraint (2T) now reads,

$$c_t^y + P_t h_{t+1}^y + P_t^r h_{t+1}^r - d_t \leq 1. \tag{8}$$

The flow of funds constraints faced by a landlord (who buys houses to rent them to the young) born at t when middle-aged and old at $t+1$ become, respectively,

$$c_{t+1}^m + \left[P_{t+1} - P_{t+1}^r\right] h_{t+1}^l + a_{t+1} \leq P_{t+1} h_{t+1}^y - R_t d_t, \tag{9}$$

and

$$c_{t+2}^o \leq (1-\tau)\, P_{t+2} h_{t+1}^l + R_{t+1} a_{t+1}, \tag{10}$$

where h_{t+1}^l is the landlord's stock of houses for rent. The parameter τ is aimed at capturing the presence of frictions in the rental market that raise the cost of renting a house.[3] In particular, the presence of τ implies that the resale value of a house that has been rented during the last period is lower than if that same house would have been occupied by the owner.

Finally, as in the model used in the main text, the landlord faces a borrowing constraint similar to (5T), i.e.

$$-a_{t+1} \leq (1-\theta)\, P_{t+1} h_{t+1}^l. \tag{11}$$

Since we focus on situations in which the rental market may not be active (i.e. zero demand or

[3]In particular, along the lines of classical references by Henderson and Ioannides (1983) and Poterba (1984), we introduce this distortion that aims to capture informational asymmetries or tax distortions that produces a bias against renting a house vis-à-vis owning it.

zero supply or both), it is helpful consider the following non-negativity constraints,

$$h^r_{t+1} \geq 0, \tag{12}$$

$$h^l_{t+1} \geq 0. \tag{13}$$

The first-order conditions of this problem are:

$$\frac{\beta}{h^y_{t+1}+h^r_{t+1}}+\frac{\beta P_{t+1}}{c^m_{t+1}} = \left[\frac{1}{c^y_t}-(1-\theta)\,\mu^y_t\right]P_t, \tag{14}$$

$$\frac{\beta}{h^y_{t+1}+h^r_{t+1}}+\varphi^r_{t+1} = \frac{P^r_t}{c^y_t}, \tag{15}$$

$$\beta\frac{P_{t+1}-P^r_t}{c^m_{t+1}} = \beta^2(1-\tau)\frac{P_{t+2}}{c^o_{t+2}}+(1-\theta)\,\mu^m_{t+1}P_{t+1}+\varphi^l_{t+1}, \tag{16}$$

$$\frac{1}{c^y_t} = \beta\frac{R_t}{c^m_{t+1}}+\mu^y_t, \tag{17}$$

$$\frac{1}{c^m_{t+1}} = \beta\frac{R_{t+1}}{c^o_{t+2}}+\mu^m_{t+1}, \tag{18}$$

$$\mu^y_t\left[(1-\theta)\,P_t h^y_{t+1}-d_t\right] = 0, \tag{19}$$

$$\mu^m_{t+1}\left[(1-\theta)\,P_{t+1}h^l_{t+1}+a_{t+1}\right] = 0, \tag{20}$$

$$h^r_{t+1}\varphi^r_{t+1} = 0, \tag{21}$$

$$h^l_{t+1}\varphi^l_{t+1} = 0, \tag{22}$$

where μ^y_t, μ^m_{t+1}, φ^r_{t+1} and φ^l_{t+1} are the Lagrange multipliers of the constraints (5T), (11), (12), and (13), respectively.

Since our objective is to analyze the effects of the rental market on the existence of stationary bubbles, we henceforth restrict our attention to the steady state version of the model developed above. The following lemma describes some key features of any steady state equilibrium with an operative rental market.

Lemma A-IV. If $\tau > 0$, then renters (*landlords*) are financially constrained (*unconstrained*) in a steady state equilibrium with an operative rental market.

Proof. We first prove that renters are constrained by contradiction. Assume that $\mu^y = 0$. Then, combining (14), (15), and (17), it follows that

$$P^r-(1-1/R)\,P=\varphi^r. \tag{23}$$

Assuming that $\varphi^l = 0$ and using (16) and (18), we can write

$$P^r-(1-1/R)\,P-\tau P/R \geq 0, \tag{24}$$

where the last expression holds as a strict inequality if $\mu^m > 0$. Thus, by combining (23) and (24), we see that $\varphi^r = \tau P/R > 0$, which according to (21) is inconsistent with a positive demand for rented houses. The intuition behind this result is the following. The user cost of purchasing a housing unit for an unconstrained homeowner is $P(1-1/R)$, while the user cost for a renter is P^r, which, according to (24) is always greater than $P(1-1/R)$ if $\tau > 0$. Then, as renters are constrained, it must be the case that landlords (at the aggregate) are net suppliers of loans and hence unconstrained (i.e. $\mu^m = 0$). This concludes the proof.

We use the results in the previous lemma A-IV to solve for the allocations of a stationary (bubble-free) equilibrium with an operative rental market, which are given below in aggregate terms,

$$C^y = \gamma, \tag{25}$$

$$C^m = \gamma\beta\frac{1-(1-\theta)R}{\theta-1/\eta}, \tag{26}$$

$$C^o = \gamma\beta^2\frac{1-(1-\theta)R}{\theta-1/\eta}R, \tag{27}$$

$$H^y = \gamma\beta(1+\beta)\frac{1}{(\theta-1/\eta)P}, \tag{28}$$

$$H^r = \gamma\beta\frac{\theta\eta-(2+\beta)}{(\theta-1/\eta)P}, \tag{29}$$

$$D = \gamma\beta(1+\beta)\frac{1-\theta}{\theta-1/\eta}. \tag{30}$$

Here, $\eta \equiv P/P^r$ is the price-rent ratio. Using $\mu^m = 0$, from (16) and (18), it follows that

$$\eta = \frac{R}{R-(1-\tau)}. \tag{31}$$

This last expression implies that the supply of houses for rent is completely elastic. Then the aggregate supply of loans, computed as the residual fraction of the middle aged's net worth (i.e. landlords) that is neither consumed nor devoted to buy houses for rent, takes the following form

$$A = \gamma\beta^2\frac{1-(1-\theta)R}{\theta-1/\eta} - \left(P-P^r\right)H^r. \tag{32}$$

The following proposition constitutes the main result of this appendix.

Proposition A-IV. Let us assume that a distorted rental market (i.e. $\tau > 0$) is established in an economy that, absent such a rental market, has a steady state equilibrium with bubbles, i.e. condition (20T) holds when $R = 1$. Then,

i) if $\tau \geq \frac{\theta}{2+\beta}$, there is no steady state equilibrium in which the rental market is operative when $R = 1$, and stationary bubbles are possible;

ii) if $\tau \in \left(\tau^*, \frac{\theta}{2+\beta}\right)$, there exists a steady state in which the rental market is operative when $R = 1$ and stationary bubbles are possible. The limiting value τ^* is given by

$$\tau^* = \frac{1 + 2(1+\beta)\theta - \sqrt{[1 + 2(1+\beta)\theta]^2 - 4(2+\beta)\theta}}{2(2+\beta)}; \tag{33}$$

ii) if $\tau < \tau^*$, then stationary bubbles are not possible.

Proof

i) Combining (29) and (31) with $R = 1$, it follows that $H^r > 0$ only if $\tau < \frac{\theta}{2+\beta}$. Otherwise, the distortion in the rental market is high enough to prevent market-clearing at a rental price P^r such that landlords and renters are willing to supply and demand houses for rent, respectively, at $R = 1$. In this case, stationary bubbles are still possible since (20T) is satisfied at $R = 1$.

ii) Let us assume that $\tau < \frac{\theta}{2+\beta}$. Then combining (29), (30), (31), and (32) we can express the excess supply of loans function as follows:

$$A - D = \frac{\gamma\beta}{\theta - \tau}\left\{1 + \left[2(1+\beta) - 1/\tau\right]\theta - (2+\beta)\tau\right\}. \tag{34}$$

we have imposed $R = 1$. This function is monotonically increasing in τ and using the previous result i), it is positive for $\tau = \frac{\theta}{2+\beta}$. Then there exists a unique $\tau < \frac{\theta}{2+\beta}$ such that $A = D$ and the rental market is operative. Such unique τ pins down the threshold value τ^*, given in (33).[4]

iii) Finally, when $\tau < \tau^*$, invoking the monotonicity of (34) with respect to τ, we learn that the condition for the existence of bubbles (20T) is not satisfied, i.e. $A - D < 0$ at $R = 1$. This completes the proof.

[4]When computing τ^* as the solution of a quadratic equation, we disregard the highest solution since it is only consistent with $\tau > 1$.