

The International Diversification Puzzle when Goods Prices are Sticky: It's Really about Exchange-Rate Hedging, not Equity Portfolios

by CHARLES ENGEL AND AKITO MATSUMOTO*

Appendix

A. Solution of the Dynamic Model

An equilibrium satisfies the first order conditions, budget constraint and market clearing conditions. First we define an equilibrium formally. Then we will list the linearized first order conditions and redefine equilibrium in linearized form.

Definition A

An *equilibrium* is a set of sequences¹ $\{C_t, L_t, W_t, \tilde{\delta}_t, \gamma_t, C_{h,t}, C_{f,t}, C_{h,t}(i), C_{f,t}(i), P_{flex,h,t}, P_{flex,f,t}, P_{preset,h,t}, P_{preset,f,t}, P_t, P_{h,t}, P_{f,t}, Q_t, V_t, H_t, R_t, R_t^H, \Pi_t, \gamma_{h,t}, \gamma_{f,t}\}_{t=1}^{\infty}$ and their foreign counterparts and $\{S_t, F_t\}$, which solves the system of 50 equations² consisting of (18), (21), (22), (43), (44), (47)-(62), and their foreign counterparts plus 3 asset markets clearing conditions,³ given stochastic sequences $\{A_t, A_t^*, M_t, M_t^*\}$ and initial conditions $A_0 = A_0^*$, $M_0 = M_0^*$, $\gamma_0 = 0$, and $\gamma_0^* = 0$.

A.1 Approximated System

In this section, we derive a log-linear version of the model, under the assumption that the stochastic driving variables (productivity and money) are lognormally distributed. Many of the equations of the model are linear in logs (without any approximation). But some of the equations in the model (the budget constraint for households, the definition of profits for the firms, and the market clearing conditions) are log-linearized around unconditional means. It is immediately apparent that

* Engel: Department of Economics, University of Wisconsin, 1180 Observatory Drive. Madison, WI 53706 (e-mail: cengel@ssc.wisc.edu); Matsumoto: Research Department, International Monetary Fund, 700 19th Street, N.W. Washington, DC 20431 (e-mail: amatsumoto@imf.org). We are very grateful to Mark Gertler, Olivier Jeanne, Karen Lewis, Maury Obstfeld, Paolo Pesenti, Cedric Tille, Eric van Wincoop, Frank Warnock, two anonymous referees, the editor, Olivier Blanchard, and seminar participants at several institutions for helpful comments. Engel acknowledges the support of the National Science Foundation through a grant to the University of Wisconsin. The views expressed in this paper are those of the authors and do not necessarily represent those of the Federal Reserve System, the IMF or IMF policy. Earlier drafts of this paper were circulated under the title "Home Bias in Equities under New Open Economy Macroeconomics", "Portfolio Choice and Home Bias in Equities in a Monetary Open-Economy DSGE Model.", and "Portfolio Choice and International Risk Sharing in a Monetary Open-Economy DSGE Model."

¹ There are $24 \times 2 + 2$ variables.

² The number of equations should be 51, but one is redundant by Walras' Law.

³ $\gamma_{h,t} + \gamma_{h,t}^* = 1$, $\gamma_{f,t} + \gamma_{f,t}^* = 1$, and $F_t \tilde{\delta}_t = \tilde{\delta}_t^*$.

our assumptions of stationary productivity processes and unit-root monetary processes imply that nominal variables have unit roots and real variables are stationary. So we log-linearize around the unconditional means of the logs of real variables.⁴

In some of the log-linearized equations below, the algebra is simplified considerably if we use the result that $\overline{p_h - p} = 0$. (In our notation, \bar{x} represents the unconditional mean of x_t .) While we could proceed with the derivations without using this result, and then verify in the solutions that this result is true, it is easier to demonstrate this first and use it in some of the log-linearizations.

First, in the definition of profits for the home firm, divide both sides of equation (62) by P_t , then evaluate the equation at the point of expansion for the log-linearization:

$$\exp(\overline{\pi - p}) = \exp(\overline{c}) \left[\frac{1}{2} \exp\left((1 - \omega)(\overline{p_h - p})\right) + \frac{1}{2} \exp\left((1 - \omega)(\overline{p_h^* - p^*})\right) \right] - \exp(\overline{w - p + \bar{l}}).$$

Here we have used symmetry to give us $\overline{c} = \overline{c^*}$ and $\overline{s + p^* - p} = 0$.

Divide the budget constraint (41) by P_t , then evaluate the equation at the point of expansion for the log-linearization:

$$\exp(\overline{\pi - p}) = \exp(\overline{c}) - \exp(\overline{w - p + \bar{l}}).$$

In deriving this expression, we have used symmetry to give us $\overline{s + q^* - q} = 0$, $\overline{s + \pi^* - \pi} = 0$, and $\overline{s - f} = 0$. We have also used $\gamma_{f,t} + \gamma_{h,t} = 1$ and $M_t = M_{t-1} + Tr_t$.

Now comparing the two equations we have derived, we must have

$$\frac{1}{2} \exp\left((1 - \omega)(\overline{p_h - p})\right) + \frac{1}{2} \exp\left((1 - \omega)(\overline{p_h^* - p^*})\right) = 1.$$

This can be written as

$$\frac{1}{2} \exp\left((1 - \omega)(\overline{p_h - p})\right) + \frac{1}{2} \exp\left(-(1 - \omega)(\overline{p_h - p})\right) = 1,$$

where we have used symmetry to give us that $\overline{p_h^* - p^*} = \overline{p_f - p}$, and linearized (18) to get $\overline{p_h - p} = -(\overline{p_f - p})$. It then follows that $\overline{p_h - p} = 0$, which is the result we will use below to simplify some of the log-linearizations.

A few more notational conventions: We denote \hat{x}_t as the deviation from the conditional mean—that is, $\hat{x}_t \equiv x_t - E_{t-1}x_t$ and $\hat{E}x_{t+s} \equiv E_t \ln x_{t+s} - E_{t-1} \ln x_{t+s}$. We will also denote the world variables as $x_t^w \equiv \frac{1}{2}x_t + \frac{1}{2}x_t^*$ and the relative variables as $x_t^R \equiv x_t - x_t^*$.

⁴ We could easily accommodate unit-root processes in productivity. Then real variables expressed in “efficiency units” would be stationary. However, there is no real gain from this generalization, so we maintain stationary productivity shocks to simplify the algebra.

A.1.1 The first order conditions for households

Suppressing constant terms and taking logs, the first order condition for consumption (54) can be written as

$$c_t = \frac{1}{\rho}(m_t - p_t). \quad (\text{A.1})$$

Using equation (A.1), equation (43) can be expressed as

$$\psi l_t = -m_t + w_t. \quad (\text{A.2})$$

Some of the equations of the model are log-linear (such as (A.1) and (A.2)), and therefore, in the presence of lognormal distributions, offer exact solutions. But others (such as the budget constraint, the market clearing condition, and the expression for a firm's profits) require approximations. Because all shocks are lognormal, the solution of the approximated model will take on a lognormal distribution. We can use equation (54) to express (44) as

$$E_{t-1}(s_t) + \frac{1}{2} \text{var}_{t-1}(s_t) - \text{cov}_{t-1}(m_t, s_t) = f_t, \quad (\text{A.3})$$

$$E_{t-1}(r_t - (m_t - m_{t-1})) - \text{cov}_{t-1}(m_t, r_t) + \frac{1}{2} \text{var}_{t-1}(r_t) + \frac{1}{2} \text{var}_{t-1}(m_t) = 0 \quad (\text{A.4})$$

$$\begin{aligned} E_{t-1}(r_t^* + s_t - s_{t-1} - (m_t - m_{t-1})) + \frac{1}{2} \text{var}_{t-1}(r_t^*) + \frac{1}{2} \text{var}_{t-1}(m_t) + \frac{1}{2} \text{var}_{t-1}(s_t) \\ - \text{cov}_{t-1}(m_t, r_t^*) + \text{cov}_{t-1}(s_t, r_t^*) - \text{cov}_{t-1}(m_t, s_t) = 0 \end{aligned} \quad (\text{A.5})$$

A.1.2 The budget constraint

We log-linearize the budget constraint (53) to get

$$\begin{aligned} p_t + c_t + \frac{\beta}{1-\beta}(1-\zeta)v_t + \frac{\beta}{1-\beta}\zeta h_t \\ = \frac{1}{1-\beta}(1-\zeta) \left[v_{t-1} + r_t - \gamma_t(r_t^R - \hat{s}_t) \right] + \frac{1}{1-\beta}\zeta(h_{t-1} + r_t^H) + \delta_t(s_t - f_t) \end{aligned} \quad (\text{A.6})$$

Here, $\zeta \equiv \frac{\exp(\overline{w-p+\bar{l}})}{\exp(\bar{c})}$, and $\delta_t \equiv \frac{\tilde{\delta}_t F_t}{M_{t-1}} \exp(\overline{m-p-\bar{c}})$. In deriving this expression, we

have used the fact that by symmetry, $\overline{v-p} = \overline{q-p}$, and then use equation (47) to derive

$\exp(\overline{q-p}) = \frac{\beta}{1-\beta} \exp(\overline{\pi-p})$. Similarly, from equation (49), we get

$\exp(\overline{h-p}) = \frac{\beta}{1-\beta} \exp(\overline{w-p+\bar{l}})$. Then, evaluating the budget constraint at the point of expansion,

we have $\exp(\bar{c}) = \exp(\overline{w-p+\bar{l}}) + \exp(\overline{\pi})$.

A.1.3 The first order conditions for firms

Firms set their prices optimally. The first order conditions can be written as

$$p_{flex,h,t} = w_t - a_t, \quad (A.7)$$

$$p_{flex,f,t} = (w_t^* - a_t^* + s_t), \quad (A.8)$$

$$p_{preset,h,t} = E_{t-1}(w_t - a_t) + \frac{1}{2} \text{var}_{t-1}(w_t - a_t) + \text{cov}_{t-1}(w_t - a_t, \tilde{d}_t + (\lambda - \omega)p_{ht} + \omega p_t + c_t) \quad (A.9)$$

$$p_{preset,h,t} = E_{t-1}(w_t - a_t - s_t) + \frac{1}{2} \text{var}_{t-1}(w_t - a_t) - \frac{1}{2} \text{var}_{t-1}(s_t) + \text{cov}_{t-1}(w_t - a_t - s_t, \tilde{d}_t + (\lambda - \omega)p_{ht} + \omega p_t + c_t^*) \quad (A.10)$$

Note that the conditional second moments in (A.9) and (A.10) are all constant over time, and will be treated as constant terms in subsequent linearizations.

Thus, the prices of each category of goods (59 and 60) can be expressed as following:

$$p_{h,t} = \tau p_{preset,h,t} + (1 - \tau) p_{flex,h,t}, \quad (A.11)$$

$$p_{f,t} = \tau p_{preset,f,t} + (1 - \tau) p_{flex,f,t}. \quad (A.12)$$

Combining these two and suppressing the constants, we get the expression for price index:

$$p_t = \frac{1}{2} p_{h,t} + \frac{1}{2} p_{f,t}. \quad (A.13)$$

A.1.4 Goods market clearing

The goods market clearing condition, equation (61) can be linearized as

$$l_t = \frac{1}{2} [-\omega(p_{h,t} - p_t) + c_t] + \frac{1}{2} [-\omega(p_{h,t}^* - p_t^*) + c_t^*] - a_t. \quad (A.14)$$

A.1.5 Other definitions

In rewriting the budget constraint (53), we introduced human capital. Linearizing (49) gives us

$$h_t = \frac{1 - \beta}{\beta} \sum_{s=1}^{\infty} E_t \beta^s (w_{t+s} + l_{t+s}). \quad (A.15)$$

Using the definition of R_t in equation (50), and the solution for Q_t in equation (47), we can write

$$r_t = (1 - \beta) E_t \left(\sum_{s=0}^{\infty} \beta^s \pi_{t+s} \right) - (1 - \beta) E_{t-1} \left(\sum_{s=0}^{\infty} \beta^s \pi_{t+s} \right) = (1 - \beta) \sum_{s=0}^{\infty} (\beta^s \hat{E}_t \pi_{t+s}). \quad (A.16)$$

The log of home firms' profits comes from linearizing (62):

$$\pi_t = \frac{1}{1-\zeta} \left[c_t^W + p_t^W + \frac{1}{2} s_t + \frac{1}{2} (1-\omega)(p_{h,t} - p_t) + \frac{1}{2} (1-\omega)(p_{h,t}^* - p_t^*) - \zeta(w_t + l_t) \right].$$

Similarly,

$$r_t^H = (1-\beta) \sum_{s=0}^{\infty} \left[\beta^s \hat{E}_t(w_{t+s} + l_{t+s}) \right]. \quad (\text{A.17})$$

A.2 Definition of Approximated Equilibrium

Definition B

An approximated equilibrium is a set of sequences $\{c_t, l_t, w_t, r_t, r_t^H, \delta_t, \gamma_t, p_t, v_t, h_t\}$ and their foreign counterparts, and $\{s_t, f_t\}$ that solve the system of equation (A.1)-(A.6), (A.14)-(A.17), and their foreign counterparts, given sequences $\{m_t, m_t^*, a_t, a_t^*\}$ and initial conditions $a_0^R = 0$, $m_0^R = 0$, and $\gamma_0 = \gamma_0^* = 0$. An approximated equilibrium is a reduced form of *Definition A*. Most omitted part can be easily verified and should not be confusing. We present the solutions for x_t and x_t^* in the form of solutions for x_t^R and x_t^W to facilitate the demonstration that these satisfy the equilibrium conditions.

A.3 Equilibrium Allocation

We conjecture that the following allocation is an equilibrium.

$$\begin{aligned} l_t^R &= \frac{\omega(1-\tau)-1}{1+\omega(1-\tau)\psi} a_t^R + \frac{\omega\tau}{1+\omega(1-\tau)\psi} \frac{\psi+1}{1+\omega\psi} E_{t-1} a_t^R \\ &= \frac{\omega(1-\tau)-1}{1+\omega(1-\tau)\psi} a_t^R + \frac{\omega\tau}{1+\omega(1-\tau)\psi} \frac{\psi+1}{1+\omega\psi} \mathcal{G}_R a_{t-1}^R \end{aligned} \quad (\text{A.18})$$

$$\begin{aligned} l_t^W &= \frac{1}{\rho+(1-\tau)\psi} \left[(1-\tau-\rho)a_t^W + \tau m_t^W + \tau E_{t-1} \left(\frac{\rho(\psi+1)}{\rho+\psi} a_t^W - m_t^W \right) \right] \\ &= \frac{1}{\rho+(1-\tau)\psi} \left\{ (1-\tau-\rho)a_t^W + \tau m_t^W + \tau \left[\frac{\rho(\psi+1)}{\rho+\psi} \mathcal{G}_W a_{t-1}^W - m_{t-1}^W \right] \right\} \end{aligned} \quad (\text{A.19})$$

$$w_t^R = \psi \left\{ \frac{(1-\tau)(\omega-1)-\tau}{1+\omega(1-\tau)\psi} a_t^R + \frac{\omega\tau}{1+\omega(1-\tau)\psi} \frac{\psi+1}{1+\omega\psi} \mathcal{G}_R a_{t-1}^R \right\} + m_t^R \quad (\text{A.20})$$

$$w_t^W = \frac{\psi}{\rho+(1-\tau)\psi} \left\{ (1-\tau-\rho)a_t^W + \tau \left[\frac{\rho(\psi+1)}{\rho+\psi} \mathcal{G}_W a_{t-1}^W - m_{t-1}^W \right] \right\} + \frac{\rho+\psi}{\rho+(1-\tau)\psi} m_t^W \quad (\text{A.21})$$

$$p_t^R = \tau m_{t-1}^R + (1-\tau)m_t^R \quad (\text{A.22})$$

$$p_t^W = -\frac{\rho\tau}{\rho+(1-\tau)\psi} \left[\frac{\rho(\psi+1)}{\rho+\psi} \mathfrak{G}_W a_{t-1}^W - m_{t-1}^W \right] - (1-\tau) \frac{\rho+\psi}{\rho+(1-\tau)\psi} \left[\frac{\rho(\psi+1)}{\rho+\psi} a_t^W - m_t^W \right] \quad (\text{A.23})$$

$$c_t^R = \frac{1}{\rho} \tau (m_t^R - m_{t-1}^R) \quad (\text{A.24})$$

$$c_t^W = \frac{\tau}{\rho+(1-\tau)\psi} \left[\frac{\rho(\psi+1)}{\rho+\psi} \mathfrak{G}_W a_{t-1}^W + (m_t^W - m_{t-1}^W) \right] + (1-\tau) \frac{1+\psi}{\rho+(1-\tau)\psi} a_t^W \quad (\text{A.25})$$

$$s_t = m_t^R. \quad (\text{A.26})$$

$$f_t = m_{t-1}^R. \quad (\text{A.27})$$

$$r_t^R = (1-\beta)(\psi+1) \left[\frac{(1-\tau)(\omega-1)-\tau}{1+\omega(1-\tau)\psi} + \frac{1}{1-\zeta} \frac{\tau}{1+\omega(1-\tau)\psi} + \frac{\omega-1}{1+\omega\psi} \frac{\beta\mathfrak{G}_R}{1-\beta\mathfrak{G}_R} \right] \hat{a}_t^R + \hat{m}_t^R \quad (\text{A.28})$$

$$r_t^W = (1-\beta)(\psi+1) \left[\frac{(1-\tau)(1-\rho)}{\rho+(1-\tau)\psi} + \frac{\zeta}{1-\zeta} \frac{\tau\rho}{\rho+(1-\tau)\psi} + \frac{1-\rho}{\rho+\psi} \frac{\beta\mathfrak{G}_W}{1-\beta\mathfrak{G}_W} \right] \hat{a}_t^W \\ + \left[1 - \frac{\zeta}{1-\zeta} \frac{(1-\beta)(\psi+1)\tau}{\rho+(1-\tau)\psi} + \frac{1}{1-\zeta} \frac{(1-\beta)(1-\rho)\tau}{\rho+(1-\tau)\psi} \right] \hat{m}_t^W \quad (\text{A.29})$$

$$r_t^{HR} = (1-\beta)(\psi+1) \left[\frac{(1-\tau)(\omega-1)-\tau}{1+\omega(1-\tau)\psi} + \frac{\omega-1}{1+\omega\psi} \frac{\beta\mathfrak{G}_R}{1-\beta\mathfrak{G}_R} \right] \hat{a}_t^R + \hat{m}_t^R \quad (\text{A.30})$$

$$r_t^{HW} = (1-\beta)(\psi+1) \left[\frac{1-\tau-\rho}{\rho+(1-\tau)\psi} + \frac{1-\rho}{\rho+\psi} \frac{\beta\mathfrak{G}_W}{1-\beta\mathfrak{G}_W} \right] \hat{a}_t^W + \left[1 + \frac{(1-\beta)(\psi+1)\tau}{\rho+(1-\tau)\psi} \right] \hat{m}_t^W \quad (\text{A.31})$$

$$\delta \equiv \delta_t = \frac{1}{2} \left(\frac{1}{\rho} - 1 \right) \tau \quad (\text{A.32})$$

$$\gamma \equiv \gamma_t = \gamma_t^* = \frac{1}{2} \frac{(\omega-1) \left[\frac{1-\tau}{1+\omega(1-\tau)\psi} + \frac{1}{1+\omega\psi} \frac{\beta\mathfrak{G}_R}{1-\beta\mathfrak{G}_R} \right]}{\frac{\tau\zeta}{1+\omega(1-\tau)\psi} + (1-\zeta)(\omega-1) \left[\frac{1-\tau}{1+\omega(1-\tau)\psi} + \frac{1}{1+\omega\psi} \frac{\beta\mathfrak{G}_R}{1-\beta\mathfrak{G}_R} \right]} \quad (\text{A.33})$$

$$h_t^R = \frac{1-\beta}{\beta} \left[(\psi+1) \frac{\omega-1}{1+\omega\psi} \frac{\beta\mathfrak{G}_R}{1-\beta\mathfrak{G}_R} a_t^R \right] + m_t^R \quad (\text{A.34})$$

$$h_t^W = \frac{1-\beta}{\beta} \left[(\psi+1) \frac{1-\rho}{\rho+\psi} \frac{\beta\mathfrak{G}_W}{1-\beta\mathfrak{G}_W} a_t^W \right] + m_t^W \quad (\text{A.35})$$

$$v_t^R = \frac{-\zeta}{1-\zeta} \frac{1-\beta}{\beta} \left[(\psi+1) \frac{\omega-1}{1+\omega\psi} \frac{\beta\mathfrak{G}_R}{1-\beta\mathfrak{G}_R} a_t^R \right] + m_t^R \quad (\text{A.36})$$

$$v_t^w = \frac{1-\beta}{\beta} \left[(\psi+1) \frac{1-\rho}{\rho+\psi} \frac{\beta \mathfrak{G}_w}{1-\beta \mathfrak{G}_w} a_t^w \right] + m_t^w \quad (\text{A.37})$$

Notice that this allocation replicates the allocation when a full set of state-contingent bonds is traded:

$$\rho(c_t - c_t^*) = s_t + p_t^* - p_t. \quad (\text{A.38})$$

A.4 Proof

We will show this allocation satisfies the equilibrium conditions.

A.4.1 Fundamental Variables

We now prove that the first order conditions for fundamental variables and labor market clearing conditions are in fact satisfied.

It is immediate to confirm that equations (A.18) – (A.21) satisfy equation (A.2). Likewise it is straightforward to check that (A.22) – (A.25) satisfy (A.1).

We can also verify that (A.18), (A.20) and (A.26) satisfy the relative version of the labor market clearing condition (A.14):

$$l_t^R = -(1-\tau)\omega(w_t^R - a_t^R - s_t) - \tau\omega E_{t-1}(w_t^R - a_t^R - s_t) - a_t^R. \quad (\text{A.39})$$

It is tedious but straightforward to verify that (A.19) and (A.21) satisfy the world version of labor market clearing condition (A.14):

$$l_t^W = c_t^W - a_t^W. \quad (\text{A.40})$$

Using equations (A.21) and (A.23), and using (A.20) and (A.26), we can show

$$p_t^W = \tau E_{t-1}(w_t^W - a_t^W) + (1-\tau)(w_t^W - a_t^W) \quad (\text{A.41})$$

$$p_t^R = \tau E_{t-1}s_t + (1-\tau)s_t \quad (\text{A.42})$$

are satisfied. Note that the variance and covariance terms in (A.9) and (A.10) are constant, from the solutions above. Substituting equations (A.7) – (A.12) into (A.13), and suppressing constant terms, we see that (A.37) and (A.38) are the solutions to the world and relative versions of (A.13).

So far, we have proved equations (A.1), (A.2), (A.13), and (A.14) are satisfied.

A.4.2 Returns on assets

In order to show that this allocation in fact satisfies the first order conditions for asset holdings, we want to calculate the rate of return on assets – human capital and equities.

Since $w_{t+s} + l_{t+s} = (\psi+1)(l_{t+s}^W + \frac{1}{2}l_{t+s}^R) + m_{t+s}^W + \frac{1}{2}m_{t+s}^R$, the return on the human capital is

$$\begin{aligned}
r_t^H &= (1-\beta) \sum_{s=0}^{\infty} \hat{E}_t \beta^s \left[(\psi+1)(l_{t+s}^W + \frac{1}{2}l_{t+s}^R) + m_{t+s}^W + \frac{1}{2}m_{t+s}^R \right] \\
&= (1-\beta)(\psi+1) \left\{ \frac{1}{\rho+(1-\tau)\psi} \left[(1-\tau-\rho)\hat{a}_t^W + \tau\hat{m}_t^W \right] \right\} \\
&\quad + (1-\beta)(\psi+1) \left\{ \frac{1-\rho}{\rho+\psi} \frac{\beta\mathcal{G}_W}{1-\beta\mathcal{G}_W} \hat{a}_t^W + \frac{1}{2} \left[\frac{(1-\tau)(\omega-1)-\tau}{1+\omega(1-\tau)\psi} + \frac{\omega-1}{1+\omega\psi} \frac{\beta\mathcal{G}_R}{1-\beta\mathcal{G}_R} \right] \hat{a}_t^R \right\} \\
&\quad + (\hat{m}_t^W + \frac{1}{2}\hat{m}_t^R).
\end{aligned} \tag{A.43}$$

Subtracting the foreign counterpart, we get equation (A.30). Adding the foreign counterpart gives us the solution to $r_t^{H^W}$.

Following similar step as in the return on human capital, we get the return on equity:

$$\begin{aligned}
r_t &= (1-\beta)(\psi+1) \left\{ \left[\frac{(1-\tau)(1-\rho)}{\rho+(1-\tau)\psi} + \frac{1}{1-\zeta} \frac{\tau\rho\zeta}{\rho+(1-\tau)\psi} + \frac{1-\rho}{\rho+\psi} \frac{\beta\mathcal{G}_W}{1-\beta\mathcal{G}_W} \right] \hat{a}_t^W \right. \\
&\quad \left. + \frac{1}{2} \left[\frac{(1-\tau)(\omega-1)-\tau}{1+\omega(1-\tau)\psi} + \frac{1}{1-\zeta} \frac{\tau}{1+\omega(1-\tau)\psi} + \frac{\omega-1}{1+\omega\psi} \frac{\beta\mathcal{G}_R}{1-\beta\mathcal{G}_R} \right] \hat{a}_t^R \right\} \\
&\quad + \left[1 - \frac{\zeta}{1-\zeta} \frac{(1-\beta)(\psi+1)\tau}{\rho+(1-\tau)\psi} + \frac{1}{1-\zeta} \frac{(1-\beta)(1-\rho)\tau}{\rho+(1-\tau)\psi} \right] \hat{m}_t^W + \frac{1}{2} \hat{m}_t^R
\end{aligned} \tag{A.44}$$

Subtracting the foreign counterpart, we get (A.28), and adding the foreign counterpart gives us that (A.29) is the solution for r_t^W . So, we have confirmed (A.16) and (A.17).

A.4.3 Asset Allocation

Since we replicate complete markets, these allocations should satisfy the first order conditions for the asset allocation as expressed in equations (44) and (55). We will prove that linearized version of them (A.3) – (A.4) are satisfied. From (A.26) and (A.27), we see $f_t = E_{t-1}s_t$. So, for equation (A.3) to be satisfied, we need

$$\text{cov}_{t-1}(m_t^R, s_t) = \text{var}_{t-1}(s_t), \tag{A.45}$$

which follows since $s_t = m_t^R$.

Since from (A.28) and (A.29), r_t is i.i.d., we have $E_{t-1}(r_t - (m_t - m_{t-1}))$ is constant.

Likewise, using (A.26), $E_{t-1}(r_t^* + s_t - s_{t-1} - (m_t - m_{t-1}))$ is constant. We can solve directly for these expectations from equations (A.4) and (A.5), using the covariances and variances implied by our solution in (A.18) – (A.33). But the following restriction links (A.4) and (A.5):

$$\text{cov}_{t-1}(-m_t, r_t) + \frac{1}{2} \text{var}_{t-1}(r_t) = \text{cov}_{t-1}(-m_t, s_t + r_t^*) + \frac{1}{2} \text{var}_{t-1}(s_t + r_t^*). \quad (\text{A.46})$$

We verify this by using $r_t = r_t^W + \frac{1}{2} r_t^R$, and rewrite (A.46) as

$$\text{cov}_{t-1}(m_t^W + \frac{1}{2} m_t^R, r_t^R - m_t^R) + \frac{1}{2} \text{var}_{t-1}(m_t^R + r_t^W - \frac{1}{2} r_t^R) - \frac{1}{2} \text{var}_{t-1}(r_t^W + \frac{1}{2} r_t^R) = 0. \quad (\text{A.47})$$

We utilize orthogonality between world shocks and relative shocks to simplify the first term:

$$\text{cov}_{t-1}(m_t^W + \frac{1}{2} m_t^R, r_t^R - m_t^R) = \frac{1}{2} \text{cov}_{t-1}(m_t^R, r_t^R - m_t^R). \quad (\text{A.48})$$

The second and third terms can be expressed as

$$\begin{aligned} \frac{1}{2} \text{var}_{t-1}(m_t^R + r_t^W - \frac{1}{2} r_t^R) - \frac{1}{2} \text{var}_{t-1}(r_t^W + \frac{1}{2} r_t^R) &= \frac{1}{2} \text{var}_{t-1}(m_t^R) - \frac{1}{2} \text{cov}_{t-1}(m_t^R, r_t^R) \\ &= -\frac{1}{2} \text{cov}_{t-1}(m_t^R, r_t^R - m_t^R) \end{aligned} \quad (\text{A.49})$$

We confirm that this allocation in fact satisfies the first order conditions for asset allocations.

So (A.3) – (A.5) are satisfied.

A4.4 Human Wealth

To verify that (A.34) and (A.35) provide the solution for human wealth (A.15), we use (A.18) – (A.21) to write

$$\begin{aligned} h_t &= \frac{1-\beta}{\beta} \sum_{s=1}^{\infty} E_t \beta^s (w_{t+s} + l_{t+s}) \\ &= \frac{1-\beta}{\beta} \sum_{s=1}^{\infty} \beta^s \left\{ (\psi + 1) \left[l_{t+s}^W + \frac{1}{2} l_{t+s}^R \right] + m_{t+s}^W + \frac{1}{2} m_{t+s}^R \right\} \\ &= \frac{1-\beta}{\beta} \sum_{s=1}^{\infty} \beta^s \left\{ (\psi + 1) \left[\frac{1-\rho}{\rho + \psi} \mathfrak{G}_W^s a_t^W + \frac{1}{2} \frac{\omega-1}{1+\omega\psi} \mathfrak{G}_R^s a_t^R \right] + m_t^W + \frac{1}{2} m_t^R \right\} \\ &= \frac{1-\beta}{\beta} \left\{ (\psi + 1) \left[\frac{1-\rho}{\rho + \psi} \frac{\beta \mathfrak{G}_W^s}{1-\beta \mathfrak{G}_W^s} a_t^W + \frac{1}{2} \frac{\omega-1}{1+\omega\psi} \frac{\beta \mathfrak{G}_R^s}{1-\beta \mathfrak{G}_R^s} a_t^R \right] + \frac{\beta}{1-\beta} \left[m_t^W + \frac{1}{2} m_t^R \right] \right\} \end{aligned} \quad (\text{A.50})$$

Then subtracting the foreign counterpart of (A.50), we get (A.34), and adding the foreign counterpart gives us (A.35).

A.4.5 Budget Constraint

First, world budget constraint expressed in home currency is the following:

$$p_t^W + c_t^W + \frac{\beta}{1-\beta} \{ (1-\zeta) v_t^W + \zeta h_t^W \} = \frac{1}{1-\beta} (1-\zeta) (r_t^W + v_{t-1}^W) + \frac{1}{1-\beta} \zeta (r_t^{H^W} + h_{t-1}^W) \quad (\text{A.51})$$

where we have used $\gamma_t = \gamma_t^*$. We have also used $\delta_t(s_t - f_t) + \delta_t^*(-s_t + f_t) = 0$, which requires $\delta_t = \delta_t^*$. This requires some explanation. The home currency earnings, expressed in home currency, from the forward market are $\tilde{\delta}_t(S_t - F_t)$. That means that the foreign currency earnings for the foreign country are $\tilde{\delta}_t\left(\frac{F_t}{S_t} - 1\right)$, which can be written as $\tilde{\delta}_t F_t\left(\frac{1}{S_t} - \frac{1}{F_t}\right)$. So, the foreign budget constraint, symmetrically to the home budget constraint, will contain the term $\tilde{\delta}_t^*\left(\frac{1}{S_t} - \frac{1}{F_t}\right)$, where $\tilde{\delta}_t^* = \tilde{\delta}_t F_t$. Using this relationship, we can establish

$$\delta_t^* = \frac{\tilde{\delta}_t^*}{F_t M_{t-1}^*} e^{\overline{m-p-\bar{c}}} = \frac{\tilde{\delta}_t F_t}{M_{t-1}} e^{\overline{m-p-\bar{c}}} = \delta_t, \quad (\text{A.52})$$

where we have used (A.27), and $\overline{m-p} = \overline{m^* - p^*}$ and $\bar{c} = \bar{c}^*$.

The world budget constraint holds with any realization of a_t^W and m_t^W since equation (A.51) simply indicates that total world wealth carried over into the next period is equal to the value of previous wealth, plus returns, less world consumption. More explicitly, because

$$v_t^W + h_t^W = \frac{1-\beta}{\beta} E_t \sum_{s=1}^{\infty} \beta^s (\pi_{t+s}^W + w_{t+s}^W + l_{t+s}^W) = \frac{1-\beta}{\beta} E_t \sum_{s=1}^{\infty} \beta^s (p_{t+s}^W + c_{t+s}^W), \quad (\text{A.53})$$

both sides of the equation are the sum of future consumption.

Finally, we examine relative budget constraint:

$$\begin{aligned} p_t^R + c_t^R - s_t + \frac{\beta}{1-\beta} [(1-\zeta)v_t^R + \zeta h_t^R - s_t] \\ = \frac{1}{1-\beta} (1-\zeta) [r_t^R - s_t + v_{t-1}^R - (\gamma_t + \gamma_t^*)(r_t^R - \hat{s}_t)] + \frac{1}{1-\beta} \zeta (r_t^{H^R} - s_t + h_{t-1}^R) + 2\delta_t \hat{s}_t \end{aligned} \quad (\text{A.54})$$

Direct substitution from the solutions verifies this equation, but it is helpful to break this down into steps.

Using $\gamma = \gamma_t = \gamma_t^*$, and the solutions for c_t^R , p_t^R , and s_t , we can write

$$\begin{aligned} \left(\frac{1}{\rho} - 1\right) \tau \hat{m}_t^R + \frac{\beta}{1-\beta} [(1-\zeta)v_t^R + \zeta h_t^R - m_t^R] \\ = \frac{1}{1-\beta} (1-\zeta) [r_t^R - \hat{m}_t^R - 2\gamma_t (r_t^R - \hat{m}_t^R)] + \frac{1}{1-\beta} \zeta (r_t^{H^R} - \hat{m}_t^R) + 2\delta_t \hat{m}_t^R \\ + \frac{1}{1-\beta} [(1-\zeta)v_{t-1}^R + \zeta h_{t-1}^R m_{t-1}^R] \end{aligned} \quad (\text{A.55})$$

Using relative returns (A.28) – (A.31), we get

$$\begin{aligned}
& \left[\left(\frac{1}{\rho} - 1 \right) \tau - 2\delta_i \right] \hat{m}_i^R + \frac{\beta}{1-\beta} \left[(1-\zeta)v_i^R + \zeta h_i^R - m_i^R \right] \\
& = \left\{ (1-2\gamma_i)(1-\zeta)(\psi+1) \left[\frac{(1-\tau)(\omega-1)-\tau}{1+\omega(1-\tau)\psi} + \frac{1}{1-\zeta} \frac{\tau}{1+\omega(1-\tau)\psi} + \frac{\omega-1}{1+\omega\psi} \frac{\beta\mathcal{G}_R}{1-\beta\mathcal{G}_R} \right] \right\} \hat{a}_i^R \quad (\text{A.56}) \\
& + \left\{ \zeta(\psi+1) \left[\frac{(1-\tau)(\omega-1)-\tau}{1+\omega(1-\tau)\psi} + \frac{\omega-1}{1+\omega\psi} \frac{\beta\mathcal{G}_R}{1-\beta\mathcal{G}_R} \right] \right\} \hat{a}_i^R + \frac{1}{1-\beta} \left[(1-\zeta)v_{i-1}^R + \zeta h_{i-1}^R - m_{i-1}^R \right]
\end{aligned}$$

By substituting expressions for δ_i and γ_i from (A.32) and (A.33). into (A.56), we get

$$\beta \left[(1-\zeta)v_i^R + \zeta h_i^R - m_i^R \right] = (1-\zeta)v_{i-1}^R + \zeta h_{i-1}^R - m_{i-1}^R. \quad (\text{A.57})$$

But (A.34) and (A.36) give us

$$(1-\zeta)v_i^R + \zeta h_i^R - m_i^R = 0, \quad (\text{A.58})$$

so (A.57) holds.

We have verified that equations (A.1)-(A.6) and (A.14)-(A.17) are satisfied.