Aggregate Implications of a Credit Crunch: Online Appendix Not for Publication

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D Analysis of Economy with CRRA Preferences and Persistent Shocks

In this appendix, we analyze the case with CRRA preferences. For sake of generality and to show that the assumption of iid shocks in the main text is not crucial for our main results, we also allow for persistence in the stochastic process of entrepreneurial productivity. In particular, we assume that in each period entrepreneurs retain their productivity with probability γ. With the complementary probability $1-\gamma$ entrepreneurs draw a new productivity from the distribution $\psi(z)$.

D.1 Characterization of Individual's Saving Problem

The value function of an entrepreneur with cash-in-hand m and ability z solve

$$V_t(m,z) = \max_{a'} \frac{(m-a')^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}\left[V_{t+1}(m_{t+1}(a',z),z')|z\right]$$

where $m_{t+1}(a',z) = \tilde{m}_{t+1}(z)a', \quad \tilde{m}_{t+1}(z) = \max\{z\pi_{t+1} - r_{t+1} - \delta, 0\}\lambda_t + 1 + r_{t+1}$.
The proof proceeds with a guess and verify strategy. Guess that the value function takes the form $V_t(m,z) = v_t(z)\frac{m^{1-\sigma}}{1-\sigma}$, and substitute into the Bellman equation.

$$v_t(z)\frac{m^{1-\sigma}}{1-\sigma} = \max_{a'} \frac{(m-a')^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}\left[v_{t+1}(z')|z\right] \frac{[\tilde{m}_{t+1}(z)a']^{1-\sigma}}{1-\sigma}$$

It will be useful to define the auxiliary variable

$$\nu_{t+1}(z) = \beta \mathbb{E}[v_{t+1}(z')|z]\tilde{m}_{t+1}(z)^{1-\sigma} \tag{1}$$

so that the Bellman equation is

$$v_t(z)\frac{m^{1-\sigma}}{1-\sigma} = \max_{a'} \ \frac{(m-a')^{1-\sigma}}{1-\sigma} + \nu_{t+1}(z)\frac{(a')^{1-\sigma}}{1-\sigma} \tag{2}$$

The first order condition is

$$(m-a')^{-\sigma} = \nu_{t+1}(z)(a')^{-\sigma}$$

or

$$a' = s_{t+1}(z)m, \quad s_{t+1}(z) \equiv \frac{1}{1+\nu_{t+1}(z)^{-1/\sigma}}$$

Consumption is

$$c = \frac{\nu_{t+1}(z)^{-1/\sigma}}{1+\nu_{t+1}(z)^{-1/\sigma}}m$$

Substituting into the Bellman equation (2) and canceling the terms involving $m^{1-\sigma}/(1-\sigma)$,

$$v_t(z) = \left(\frac{\nu_{t+1}(z)^{-1/\sigma}}{1+\nu_{t+1}(z)^{-1/\sigma}}\right)^{1-\sigma} + \nu_{t+1}(z)\left(\frac{1}{1+\nu_{t+1}(z)^{-1/\sigma}}\right)^{1-\sigma}$$

which after some manipulation becomes

$$v_t(z) = \left(1+\nu_{t+1}(z)^{1/\sigma}\right)^{\sigma}$$

or using the definition of $\nu_{t+1}(z)$ in (1),

$$v_t(z) = \left(1 + \left\{\beta\mathbb{E}\left[v_{t+1}(z')|z\right]\tilde{m}_{t+1}(z)^{1-\sigma}\right\}^{1/\sigma}\right)^{\sigma}$$

This is a functional equation in $v_t(z)$ that can be solved numerically.

D.2 Evolution of the Wealth Density, Aggregate Capital and Productivity

The evolution of the wealth density $\xi_t(z)$ is described by the following functional equation

$$\xi_{t+1}(z) = \frac{K_t}{K_{t+1}}\left[\gamma s_{t+1}(z)\tilde{m}_t(z)\xi_t(z) + (1-\gamma)\psi(z)s_{t+1}(z)\int \tilde{m}_t(z_{\text{-}1})\xi(z_{\text{-}1})dz_{\text{-}1}\right] \tag{3}$$

Using Lemma 4 and integrating over all z we obtain a law of motion for aggregate capital

$$K_{t+1} = \gamma K_t \int s_{t+1}(z)\tilde{m}_t(z)\xi_t(z)dz + (1-\gamma)\bar{s}_{t+1}\left[\alpha Y_t + (1-\delta)K_t\right]. \tag{4}$$

There are two cases for which the model allows for a simple aggregation, given the evolution of aggregate productivity Z_t. First, if we assume that entrepreneurs' productivity is iid over time, equation C.2 specializes to

$$K_{t+1} = \bar{s}_{t+1}\left[\alpha Y_t + (1-\delta)K_t\right].$$

The second correspond to the case of log preferences. Using that $s_{t+1}(z) = \bar{s}_{t+1} = \beta$ and applying Lemma 4 to the first term in the right hand side of equation C.2 we obtain a simple equation describing the evolution of aggregate capital:

$$K_{t+1} = \beta \left[\alpha Y_t + (1 - \delta) K_t \right].$$

While we can aggregate the model given the evolution of aggregate productivity, in the more general model the evolution of aggregate productivity is itself a function of the wealth density. Defining

$$\Xi(z) \equiv \int_0^z \xi(x) dx,$$

aggregate productivity is a capital weighted average of entrepreneurs' productivity

$$Z_t = \left(\frac{\int_{\underline{z}_t}^{\infty} z \xi(z) dz}{1 - \Xi(\underline{z}_t)} \right)^{\alpha}.$$

Finally, the cutoff is defined by

$$\lambda_{t-1}(1 - \Xi(\underline{z}_t)) = 1.$$

E Generalized Nash Bargaining: Entrepreneur-Specific Wage

Instead of a common wage, we could have worked with entrepreneur-specific wages that are determined by Nash bargaining. We here derive these wages for completeness. The steps described here follow Shimer (2010). We modify his derivations to allow for heterogeneity on the side of employers. Let $V_l(l_{it}, d_{it}, \omega_{it}, t)$ denote the marginal utility for entrepreneur i with employment l_{it}, debt d_{it}, and recruitment cost, ω_{it} of employing a worker at wage w_{it}. Let $W_i(\{l_{it}\}, t)$ denote the marginal utility for workers at the equilibrium level of employment of having one worker employed at a wage w_{it} in period t rather than unemployed.[1].

Consider first the value of an entrepreneur which is given by

$$\begin{aligned} V(l, d, \omega) = \max_{c,x,d'} \log c + \beta \mathbb{E} V(l', d', \omega') \quad \text{s.t.} \\ c + \omega x - d' = Al - w_t l - (1 + r_t) d, \\ l' = (1 - \delta) l + x, \quad x \geq 0, \quad d' \leq \phi A l' \end{aligned}$$

The first order condition for recruiting, x_{it}, is

$$\omega_{it} \frac{1}{c_{it}} = \beta \mathbb{E} V_l(l_{it+1}, d_{it+1}, \omega_{it+1}, t + 1) \tag{5}$$

[1]As shown below, this value depends on the entire distribution of employment, $\{l_{it}\}$

and the envelope condition

$$V_l(l_{it}, d_{it}, \omega_{it}, t) = \frac{A_t + (1-\delta)\omega_{it} - w_{it}}{c_{it}} \tag{6}$$

This is the marginal value to an entrepreneur of having an extra worker paid w_{it}

Next, consider workers. Workers take as given the distribution of employment and its evolution of employment. In particular, they take as given the (exogenous) job separation rate δ and the (endogenous) probability of finding a job at firm i, f_{it}. This job finding rate is defined by the requirement that the number of workers finding jobs, $f_{it}(1 - L_t)$, is equal to the number of workers recruited by firms x_{it} and hence $f_{it} = x_{it}/(1 - L_t)$. From the point of view of workers employment then evolves as $l_{it+1} = (1-\delta)l_{it} + f_{it}(1 - L_t)$. The value of a worker can then be written in recursive form as

$$W(\{l_{it}\}, t) = \log\left(\int w_{it} l_{it} di\right) - \gamma \int l_{it} di + \beta \mathbb{E} W(\{l_{it+1}\}, t+1)$$

The envelope condition is

$$W_i(\{l_{it}\}, t) = \frac{w_{it}}{C_t^W} - \gamma + \beta(1-\delta)\mathbb{E}W_i(\{l_{it+1}\}, t+1) - \beta \int f_{jt} \mathbb{E} W_j(\{l_{it+1}\}, t+1) dj \tag{7}$$

Following the same analysis as in Shimer (2010), it can easily be shown that if wages are determined by generalized Nash bargaining, the entrepreneur-specific wage w_{it} satisfies

$$(1-\phi) W_i(\{l_{it}\}, t) C_t^W = \phi V_l(l_{it}, d_{it}, \omega_{it}, t) c_{it} \tag{8}$$

where $\phi \in [0, 1]$ represents the worker's bargaining power. Multiply (7) by $(1-\phi)C_t^W$ to obtain

$$(1-\phi)W_i(\{l_{it}\}, t)C_t^W = (1-\phi)(w_{it} - \gamma C_t^W)$$
$$+\frac{C_t^W}{C_{t+1}^W}\left[\beta(1-\delta)C_{t+1}^W(1-\phi)\mathbb{E}W_i(\{l_{it+1}\}, t+1) - \beta\int f_{jt} C_{t+1}^W (1-\phi)\mathbb{E}W_j(\{l_{it+1}\}, t+1)dj\right]$$

Substitute in from (8)

$$\phi V_l(l_{it}, d_{it}, \omega_{it}, t)c_{it} = (1-\phi)(w_{it} - \gamma C_t^W)$$
$$+\frac{C_t^W}{C_{t+1}^W}\left[\beta(1-\delta)\phi\mathbb{E}V_l(l_{it+1}, d_{it+1}, \omega_{it+1}, t+1)c_{it+1} - \beta\int f_{jt}\phi\mathbb{E}V_l(l_{jt+1}, d_{jt+1}, \omega_{jt+1}, t+1)c_{jt+1}dj\right]$$

From (5)

$$\mathbb{E}V_l(l_{it+1}, d_{it+1}, \omega_{it+1}, t+1)c_{it+1} = \frac{c_{it+1}}{c_{it}}\omega_{it}$$

to eliminate $V_l(l_{it+1}, d_{it+1}, \omega_{it+1}, t+1)$ and (6) to eliminate $V_l(l_{it}, d_{it}, \omega_{it}, t)$,

$$\phi\left[A + (1-\delta)\omega_{it} - w_{it}\right] = (1-\phi)(w_{it} - \gamma C_t^W) + \frac{C_t^W}{C_{t+1}^W}\phi\left[(1-\delta)\frac{c_{it+1}}{c_{it}}\omega_{it} - \int \frac{c_{jt+1}}{c_{jt}} f_{jt}\omega_{jt} dj\right]$$

Rearranging

$$\phi\left[A+(1-\delta)\omega_{it}\left(1-\frac{C_t^W}{C_{t+1}^W}\frac{c_{it+1}}{c_{it}}\right)+\frac{C_t^W}{C_{t+1}^W}\int\frac{c_{jt+1}}{c_{jt}}f_{jt}\omega_{jt}dj-w_{it}\right]=(1-\phi)(w_{it}-\gamma C_t^W)$$

And hence

$$w_{it}=\phi\left(A+(1-\delta)\omega_{it}\left(1-\frac{C_t^W}{C_{t+1}^W}\frac{c_{it+1}}{c_{it}}\right)+\frac{C_t^W}{C_{t+1}^W}\int\frac{c_{jt+1}}{c_{jt}}f_{jt}\omega_{jt}dj\right)+(1-\phi)\gamma C_t^W$$

This is the Nash-bargaining solution. Note that wages are entrepreneur-specific because of heterogeneity in recruitment costs, and also because financing constraints imply that consumption growth rates differ across entrepreneurs.[2]

F Behavior of External Finance Relative to GDP

This section derives the effect of a credit crunch in period 1 on the external finance to GDP ratio in period 2 (the first period where a credit crunch has an effect on the economy), and contrasts it with the effect of a pure TFP shock in period 2.

Proposition 1 *The elasticities of the external finance to GDP ratio in period* 2, D_2/Y_2, *with respect to a change in TFP in period* 2, Z_2, *and to change in the tightness of collateral constraints in period* 1, λ_1 *are given by*

$$\frac{\partial\log\left(\frac{D_2}{Y_2}\right)}{\partial\log Z_2}=-1<\frac{1}{\lambda_1-1}\left(\frac{1}{\lambda_1}-\left(1-\frac{1}{\lambda_1}\right)\alpha\left(1-\frac{\underline{z}_2}{\frac{\int_{\underline{z}_2}^{\infty}z\psi(z)dz}{1-\Psi(\underline{z}_2)}}\right)\right)=\frac{\partial\log\left(\frac{D_2}{Y_2}\right)}{\partial\log\lambda_1}.$$

A credit crunch (decline in λ_1*) results in a decrease in* D_2/Y_2 *if the elasticity of* D_2/Y_2 *with respect to* λ_1 *is positive. This is the case if* $\lambda_1=1/(1-\theta_1)$ *is small, or equivalently* θ_1 *is small.*

Proof Consider first the elasticity of external finance to GDP with respect to an exogenous change in TFP. The debt to GDP ratio is $D_2/Y_2=(D_2/K_2)(K_2/Y_2)$. Further $D_2/K_2=\theta_1$ is constant, GDP is $Y_2=Z_2K_2^{\alpha}L^{1-\alpha}$ and, on impact, the economy's aggregate capital stock K_2 is fixed. Therefore it is straightforward to see that this elasticity equals

$$\frac{\partial\log\left(\frac{D_2}{Y_2}\right)}{\partial\log Z_2}=\frac{\partial\log\left(\frac{K_2}{Y_2}\right)}{\partial\log Z_2}=-1.$$

[2]Without heterogeneity and with perfect financial markets (implying $c_{it+1}/c_{it}=C_{t+1}^W/C_t^W$), the wage would simply be $w_t=\phi\left(A+f\omega\right)+(1-\phi)\gamma C_t^W$.

Next, consider the elasticity of external finance to GDP with respect to to λ_1. We have that

$$\begin{aligned} \frac{D_2}{Y_2} &= \frac{\int_{z_{i,2} \geq \underline{z}_t} d_{i,2} di}{Y_2} \\ &= \frac{(\lambda_1 - 1)(1 - \Psi(\underline{z}_2)) K_2}{Z_2 K_2^{\alpha} L^{1-\alpha}} \end{aligned}$$

Therefore

$$\frac{\partial \left(\frac{D_2}{Y_2}\right)}{\partial \lambda_1} = \left(\frac{K_2}{L}\right)^{1-\alpha} \frac{1}{Z_2} \left[1 - \Psi(\underline{z}_2) - (\lambda_1 - 1)\psi(\underline{z}_2)\frac{\partial \underline{z}_2}{\partial \lambda_1} - \frac{(\lambda_1 - 1)(1 - \Psi(\underline{z}_2))}{Z_2} \frac{\partial Z_2}{\partial \lambda_1}\right] \quad (9)$$

Differentiating $Z_2 = \left(\frac{\int_{\underline{z}_2}^{\infty} z\psi(z)dz}{1-\Psi(\underline{z}_2)}\right)^{\alpha}$ with respect to λ_1

$$\frac{\partial Z_2}{\partial \lambda_1} = \alpha \left(\frac{\int_{\underline{z}_2}^{\infty} z\psi(z)dz}{1 - \Psi(\underline{z}_2)}\right)^{\alpha-1} \frac{\psi(\underline{z}_2)}{1 - \Psi(\underline{z}_2)} \left[\frac{\int_{\underline{z}_2}^{\infty} z\psi(z)dz}{1 - \Psi(\underline{z}_2)} - \underline{z}_2\right] \frac{\partial \underline{z}_2}{\partial \lambda_1}$$

Applying the Implicit Function Theorem to the condition relating the cutoff to the quality of financial markets, $\lambda_1 (1 - \Psi(\underline{z}_2)) = 1$

$$\frac{\partial \underline{z}_2}{\partial \lambda_1} = \frac{1 - \Psi(\underline{z}_2)}{\lambda_1 \psi(\underline{z}_2)}$$

Substituting the expressions for $\partial Z_2/\partial \lambda_1$ and $\partial \underline{z}_2/\partial \lambda_1$ into (9)

$$\frac{\partial \left(\frac{D_2}{Y_2}\right)}{\partial \lambda_1} \frac{\lambda_1}{\frac{D_2}{Y_2}} = \frac{1}{\lambda_1 - 1} \left(\frac{1}{\lambda_1} - \left(1 - \frac{1}{\lambda_1}\right) \alpha \left(1 - \frac{\underline{z}_2}{\frac{\int_{\underline{z}_2}^{\infty} z\psi(z)dz}{1-\Psi(\underline{z}_2)}}\right)\right) . \square$$

References

Shimer, Robert. 2010. *Labor Markets and Business Cycles.* Princeton University Press.