Conventional and Unconventional Monetary Policy with Endogenous Collateral Constraints

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ONLINE APPENDIX

A Proofs of Lemmas and Propositions in the Text

A.1 The Irrelevance of Asset 1

As remarked in the text, one can show quite generally that the market for "asset 1" (the private debt contract which is so poorly collateralized that the value of the collateral allows repayment in full only in state $s = 1$, that is, only in the state in which the period-1 price of the durable good achieves its maximum value) is redundant.

Lemma 6 *Consider any equilibrium of any economy $\mathcal{E}$. Then either the market for asset 1 (private debt contracts which are so poorly collateralized that they default in all states but the one in which the durable good has its highest value) is* inactive*, in the sense that zero units of this security are issued in equilibrium; or it is* inessential*, in the sense that the same allocation of resources and same prices for all goods and assets could also be obtained as an equilibrium if the market were closed (i.e., if all households were subject to the additional constraint that they must choose $\psi_1^h = \varphi_1^h = 0$.*

Proof. The state-contingent payoffs on a unit of asset 1 are equivalent to those on C_1 units of the durable good (after the period 0 service flow). A household will therefore be unwilling to purchase any units of asset 1 at any price higher than $(p_3-p_2)C_1$, since C_1 units of the durable could be purchased at that price, yielding the same period 1 state-contingent return *and* relaxing the household's collateral constraint as well. Moreover, no household will be willing to purchase any units when $q_1 = (p_3 - p_2)C_1$ exactly, either, except if the household's collateral constraint does not bind.

On the other hand, an issuer must hold C_1 units of the durable in order to issue a unit of asset 1, and surrenders the durable in all states in period 1. (Technically, there need be no default in state 1, the state in which the durable is most valuable; but the issuer must pay the holder of the security an amount that is as costly as surrender of the durable in that state as well.) Hence the issuer obtains no income in any state in period 1 from the transaction, and so will not be willing to issue the security at any price less than $(p_3 - p_2)C_1$, the cost of the collateral that will then have to be surrendered.

It follows that asset 1 cannot be issued and held, in equilibrium, unless $q_1 = (p_3 - p_2)C_1$, *and* the households that hold asset 1 would obtain no value from a relaxation of their collateral constraints. But then the same equilibrium (same allocation $\overline{x}$ and same prices $(\overline{p}, \overline{q})$) can be obtained if the market for asset 1 is closed: the issuers of asset 1 could simply sell the collateral (after collecting the period 0 rental income from it) to the buyers instead, rather than using the collateral to back issuance of asset 1. The issuers should be indifferent between this plan and issuance of asset 1, since they must surrender the durable in all states in period 1 anyway, and since they would obtain the same sale price in period 0. The buyers should be indifferent as well, as they obtain an asset with the same state-contingent returns in period 1, pay the same price in period 0, and do not care about the fact that acquiring the durable relaxes their collateral constraints. Hence if an equilibrium exists in which asset 1 is issued, the existence of this market is inessential. □

The model is one that allows, in general, for the coexistence of multiple types of privately issued debt that default with different probabilities (and hence promise different rates of interest, conditional upon repayment, as well). But Lemma 6 shows that *more than two* states in period 1 are necessary in order for default to occur in equilibrium, at least on securities the existence of which matters for the character of equilibrium. We nonetheless find it convenient to study mainly examples with only two states in this paper. This means that the occurrence of default in equilibrium is not essential for the type of financial frictions with which we are concerned. In the equilibria that we study, the *possibility* of default can lead to distortions of the equilibrium allocation of resources, even though in most of our examples no default actually occurs.

A.2 Proof of Lemma 1

The period-1 return on one unit of asset S and on $1/(1+i)$ units of reserves are identical: in either case, the holder obtains one unit of money in period 1, in every possible state s. (Recall that $C_S = 1/p_{2S} \geq 1/p_{2s}\ \forall s \in \mathcal{S}$. Thus private borrowing under a contract of type S is sufficiently collateralized to be perfectly safe.) Since a positive quantity of reserves earning the interest rate i must be held in equilibrium, it is necessary that $q_S \geq 1/(1+i)$; otherwise, reserves would not be held. But if $q_S > 1/(1+i)$, no household will choose to hold asset S, since reserves are a perfect substitute available more cheaply; hence this would have to be an equilibrium with no issuance of asset S, and one in which the market for asset S is inessential.

One can also show that no equilibrium of the latter sort is possible if at least one household type has excess collateral. For suppose that $q_S > 1/(1+i)$. Then any household can obtain an arbitrage profit, relaxing its budget constraint in period 0, by increasing μ^h by a quantity $\epsilon > 0$ and issuing ϵ units of asset S. (This would result in no change in its period 1 budget in any state of the world, but increase the amount that it can spend on either non-durable consumption or rental of durable goods in period 0, given that the proceeds of issuance of the riskless debt would exceed the addition to its holdings of riskless assets.) Each household must, under an optimal plan, exploit this opportunity to the greatest extent allowed by the collateral constraint. If there exists any household with a collateral constraint that remains slack, no such opportunity must exist, and hence $q_S = 1/(1+i)$ exactly. □

A.3 Proof of Proposition 1

One observes that the household problem (1.6) can be written entirely in terms of choice variables $x^h, x_3^h, \psi^h, \varphi^h, \mu^h$; endowments $e_1^h, e_3^h, e_{s1}^h, d^h$; the collateral requirements $\{C_j\}$; period 0 *relative* prices $p_2/p_1, p_3/p_1, q/p_1$; period 1 prices; and the quantity $(1+i)p_1$, but not p_1 or i individually. The requirements for equilibrium can also be written entirely in terms of these variables.

Consider now any equilibrium associated with a given value of i. Associated with this equilibrium are particular values for each of the variables listed in the previous paragraph. If now i is varied (to some other non-negative value), these same values for each of those variables will continue to constitute an equilibrium. Note however that constancy of $(1+i)p_1$ requires p_1 to vary inversely with $(1+i)$.

Hence an equilibrium exists for arbitrary $i \geq 0$ with the properties stated in the proposition. Because p_1 varies inversely with $1+i$, and the relative prices $p_2/p_1, p_3/p_1$, q_j/p_1 are invariant, all period 0 prices must vary inversely with $1+i$. □

A.4 Proof of Proposition 2

One observes (as in the proof of Proposition 1) that both the household problem (1.6) and the requirements for equilibrium in Definition 1 can be written entirely in terms of quantities that make no reference to either d^{CB} or M. Hence equilibrium values for the set of quantities referred to in Definition 1 continue to represent equilibrium values in the event of a change in d^{CB} and corresponding change in M (so that (1.3) continues to be satisfied). Note that the equilibrium values of households' post-trade holdings of government debt and money in period 0 (for which we have not even introduced notation) are indeterminate; only the sum of these two quantities, μ^h, has a determinate equilibrium value. In the event of an open-market operation of the kind contemplated in the proposition, at least some households must change the composition of their portfolios as between their holdings of government debt and money, but the equilibrium values $\{\bar{\mu}^h\}$ do not change, and no changes in other quantities are required to induce the households to change their portfolios along a dimension on which their choice was in any event indeterminate.□

A.5 Proof of Proposition 3

Consider any value of ω satisfying (2.1). (Note that the assumption that each household holds excess collateral implies that there is an open interval of such values.[38]) Suppose that prices continue to be given by $(\overline{p}, \overline{q}$, and that the collateral requirements continue to be given by $\overline{C}$.

Then for each household h, it is possible to achieve the same consumption plan $\overline{x}^h$ as before, with a portfolio plan that is the same as before, except that now

$$x_3^h = \overline{x}_3^h - \theta^h(\omega - \bar{\omega})e_3 \tag{A.1}$$

and

$$\mu^h = \overline{\mu}^h + (1+i)(\overline{p}_3 - \overline{p}_2)\theta^h(\omega - \bar{\omega})e_3. \tag{A.2}$$

Condition (2.1) guarantees that the right-hand side of (A.1) is non-negative, while the assumption that $\omega > \bar{\omega}$ guarantees that the right-hand side of (A.2) is non-negative as well; thus these stipulations remain consistent with the non-negativity constraints on the household's portfolio. Substitution of the proposed consumption and portfolio plan into the budget constraints verifies that each is still satisfied. And finally, condition (2.1) guarantees that the household's collateral constraint continues to be satisfied. Hence the proposed plan is feasible for each household h.

One can further show that the proposed plan is not only feasible for household h, but optimal. This requires that we show that no consumption plan preferable to $\overline{x}^h$ is attainable. Consider any consumption plan $\tilde{x}^h$ such that $u^h(\tilde{x}^h) > u^h(\overline{x}^h)$. It follows that for any convex combination

$$\hat{x}^h \equiv (1-\lambda)\overline{x}^h + \lambda\tilde{x}^h,$$

where $0 < \lambda < 1$, $\hat{x}^h$ will also be strictly preferred to $\overline{x}^h$, given the quasi-concavity of preferences.

Now suppose that consumption plan $\tilde{x}^h$ is attainable, that is, that there exists a plan $(\tilde{x}^h, \tilde{\psi}^h, \tilde{\varphi}^h, \tilde{\mu}^h, \tilde{x}_3^h)$ that is consistent with all of the household's constraints in the case of policy ω. But then a plan identical to this, except with

$$x_3^h = \tilde{x}_3^h + \theta^h(\omega - \bar{\omega})e_3,$$

$$\overline{q}_S\varphi_S^h = \overline{q}_S\tilde{\varphi}_S^h + (\overline{p}_3 - \overline{p}_2)\theta^h(\omega - \bar{\omega})e_3,$$

would satisfy the household's budget constraints in both period 0 and period 1, under the original policy $\bar{\omega}$. (Here we use Lemma 1 to show that the period 1 budget

[38]Here we rely on the assumption of only a finite number of household types. In the case of an infinite number of household types, it would be necessary to strengthen the hypothesis of the proposition to require the existence of a positive lower bound for the right-hand-side of (2.1).

constraint is satisfied in each state.) Given that $\omega > \bar{\omega}$, this plan obviously satisfies all non-negativity constraints as well.

Moreover, because of the convexity of the constraint set, any convex combination of the optimal plan under policy $\bar{\omega}$ and this plan will also satisfy the household's budget constraints in both periods, and will satisfy all non-negativity constraints. And since the collateral constraint is (by hypothesis) a strict inequality under the optimal plan, there exists some sufficiently small $\lambda > 0$ for which the convex combination of the plans will also satisfy the collateral constraint. Hence the convex combination plan satisfies all of the household's constraints, under the original policy $\bar{\omega}$.

This would imply that the convex combination consumption plan $\hat{x}^h$ is attainable under the policy $\bar{\omega}$. But since $\hat{x}^h$ is strictly preferable to $\overline{x}^h$, this contradicts the assumption that the household's behavior in the equilibrium associated with policy $\bar{\omega}$ is optimal. Hence we may conclude that the plan described by (A.1)–(A.2) is optimal, under the unchanged prices $(\overline{p}, \overline{q})$ and collateral requirements $\overline{C}$.

One can further show that this collection of plans for the households implies market clearing. Note that (A.1) and (A.2) imply that

$$\sum_h x_3^h = \sum_h \overline{x}_3^h - (\omega - \bar{\omega})e_3 = (1-\omega)e_3,$$

$$\sum_h \mu^h = \sum_h \overline{\mu}^h + (\overline{p}_3 - \overline{p}_2)(\omega - \bar{\omega})e_3 = d + (1+i)(\overline{p}_3 - \overline{p}_2)\omega e_3,$$

so that conditions (iv) and (viii) of Definition 1 are satisfied. The other market-clearing conditions are unchanged by the change in ω. Finally, condition (ix) of Definition 1 continues to be satisfied, since neither the prices nor the collateral requirements have changed. Hence all requirements for equilibrium are satisfied. $\square$

A.6 Proof of Proposition 4

If all households have the same preferences and endowments, the household problem (1) is the same for each of them. Then because the household's budget set is convex and the common preferences are assumed to be strictly quasi-concave, there is a unique optimal consumption plan that solves this problem for any prices (p, q) and collateral requirements C, though the associated portfolio plan may be indeterminate. It follows that each household necessarily chooses the same consumption plan in equilibrium. Market clearing is then only possible if each household chooses to consume exactly its share of the aggregate endowment. Hence the equilibrium allocation of resources must be given by

$$\overline{x}_1^h = e_1^*, \qquad \overline{x}_2^h = e_3^*, \qquad \overline{x}_{s1}^h = e_{s1}^* \;\forall s, \qquad \overline{x}_{s2}^h = e_3^* \;\forall s$$

for each $h \in \mathcal{H}$, where stars indicated the common endowments of each of the goods.

This plan can be seen to be consistent with the household's budget constraints if the household's portfolio plan satisfies

$$\overline{x}_3^h = (1-\omega)e_3^*, \qquad \overline{\mu}^h = d^* + (1+\bar{\imath})(\overline{p}_3 - \overline{p}_2)\omega e_3^* = \mu/H,$$

$$\overline{\psi}_j^h = \overline{\varphi}_j^h \geq 0 \;\forall j, \qquad \sum_j (\overline{\varphi}_j^h / \overline{p}_{j2}) \leq (1-\omega)e_3^*. \tag{A.3}$$

One possible way to satisfy all requirements of (A.3) is by choosing $\psi_j^h = \varphi_j^h = 0 \;\forall j$, though this is not the unique solution; thus these conditions can be satisfied. Moreover, any specification of portfolio plans for the households that satisfy the above conditions will satisfy all market-clearing conditions for assets.

It remains only to show that there exist prices under which it will be optimal for a household to choose the feasible plan described above. The prices required can then be determined from the household's marginal rates of substitution, evaluated at this consumption plan. They are in fact the prices associated with an A-D equilibrium. Since the consumption plan $\overline{x}^h$ specified above is the optimal element of the A-D budget set defined by these prices, and the budget set in our model is a proper subset of the A-D budget set, the plan (which is also feasible in our model) must be the optimal element of the budget set in our model as well. Hence we have described an equilibrium.

If the equilibrium is supported by a portfolio plan for each household h in which $\psi_j^h = \varphi_j^h = 0 \;\forall j$, then since $(1-\omega)e_3^* > 0$, the collateral constraint is a strict inequality for each household. This establishes the existence of an equilibrium in which each household holds excess collateral. We can also have equilibria in which one or more households is both an issuer and a purchaser (in equal quantities) of private debt securities, to such an extent as to use all of its available collateral in issuing such securities. (There is no economic motive for a household to do so, but no penalty either, given that we abstract from transactions costs in our model.) But even in such a case, the collateral constraint could actually be tightened without requiring the household to change its consumption allocation, or to change its behavior in any way that interferes with market clearing. Thus even if the hypothesis of Proposition 3 would technically not be satisfied in such a case, the conclusion could still be established. □

A.7 Proof of Lemma 2

The homotheticity of the aggregator function $v(x_1, x_2)$ implies that in any period, and any state of the world, the optimal relative consumption x_1/x_2 is independent of the scale of the household's expenditure in that state, and is given by $x_1^h/x_2^h = r(p_2/p_1)$, where the function $r(p_2/p_1)$ is implicitly defined by

$$\frac{v_2(1,r)}{v_1(1,r)} = \frac{p_2}{p_1}.$$

Since each household's demands are in this proportion, so must be the *aggregate* demands for the two goods. Market clearing requires that the ratio of aggregate demands equal the ratio of aggregate supplies; hence the equilibrium relative price must be given by equation (2.2) in the text. □

A.8 Proof of Lemma 3

In the case that there are only two possible states in period 1, the number of types of private debt securities that we must consider can be reduced to two, as discussed in section 1.2. Moreover, the market for asset 1 is inessential, as shown by Lemma 6; so we can economize on notation by eliminating the market for this asset. There is then only one kind of private debt: riskless (fully collateralized) private debt (asset 2).[39] By Lemma 1, this must be equivalent to riskless government debt (or central-bank reserves), in any equilibrium where it is actually issued.

There are thus only two independent ways in which a household can shift income between period 0 and period 1: either by holding or issuing riskless claims (where it does not matter whether government-supplied riskless assets or privately-issue riskless debt is held), or by holding durable goods. A household can hold arbitrary positive quantities of these two types of assets (subject to the constraint that period 0 expenditure must be non-negative), but is limited in the extent to which it can hold a net *negative* position of either type. It cannot short the durable good at all; issuance of "asset 1" would amount to sale of a security that has the same state-contingent payoffs as the durable good, but the collateral constraint implies that a household that issues asset 1 must hold an equivalent quantity of the durable as collateral, so that it is not able to achieve a net negative position in assets with this pattern of returns. It can take a short position in the riskless asset, but the size of this is subject to a limit proportional to its holdings of the durable (because of the collateral requirement for issuing riskless debt).

The two dimensions of variation in the vector of intertemporal transfers $\mathbf{y}^h$ thus correspond to variation in the size of the household's effective position in the risky durable and variation in the size of its net holdings of the riskless asset. There is a unique combination of riskless assets and durables that must be held to achieve a given vector $\mathbf{y}^h$; hence given the market prices of the two types of assets, we can

[39] Fostel and Geanakoplos (2013) similarly establish that markets for risky collateralized debt are inessential, in the case that there are only two possible states in the second period. Note that this would not generally be true in the case of more than two states.

assign a well-defined cost (in terms of reduced period-0 expenditure) of any choice of y^h. This cost will be a linear function $\mathbf{a}'\mathbf{y}^h$, where $\mathbf{a}$ is a vector of state prices, defined as the two quantities $a_1, a_2 > 0$ that satisfy (2.6)–(2.7).

The constraints on a household's ability to choose a given vector of transfers $\mathbf{y}^h$ result not only from the market prices of assets, though, but also from the lower bounds on its net asset positions just discussed. The fact that the durable (the only asset that pays more, in nominal terms, in state 1 than in state 2) cannot be shorted means that $p_{11}y_1^h$ must be at least as large as $p_{21}y_2^h$ for any household. And y_2^h must be non-negative, since the collateral constraint requires a household to hold durables that are worth at least as much in state 2 as the face value of any riskless debt issued by the household. Subject to these two inequalities, however, any vector $\mathbf{y}^h$ is attainable if the household is willing (and able) to reduce period 0 expenditure by $\mathbf{a}'\mathbf{y}^h$ to pay for it.□

A.9 Proof of Lemma 4

The fact that constraint (2.12) does not bind implies that in equilibrium, $U_1^h = 0$ for all $h \in \mathcal{H}$, where we use the notation U_s^h for the partial derivative of the indirect utility function $U^h(\tilde{\mathbf{y}}^h; \mathbf{a})$ with respect to $\tilde{y}_s^h$, evaluated for the equilibrium state prices $\bar{a}$. This implies that

$$\frac{\tilde{u}_1'(c_1^h)}{\tilde{u}'(c^h)} = 2a_1$$

for all $h \in \mathcal{H}$.

Condition (3.1) then implies that

$$\frac{\alpha_1}{\alpha}\left(\frac{c_1^h}{c^h}\right)^{-\gamma} = 2a_1,$$

so that the expenditure ratio c_1^h/c^h must be the same for all households. But the aggregate expenditure ratio must equal the ratio of the values of the aggregate endowments in the two states; hence the expenditure ratio for each household must equal the ratio of the endowments. Substitution of the aggregate endowments into the above condition to determine each household's expenditure ratio then yields

$$\frac{\alpha_1}{\alpha}\left[\frac{e_{11} + (p_{12}/p_{11})e_3}{e_1 + (p_2/p_1)e_3}\right]^{-\gamma} = 2a_1.$$

This condition can be solved for the equilibrium value of the state price, $\bar{a}_1$, yielding the expression given in the lemma.□

A.10 Proof of Proposition 5

By Lemma 4, the equilibrium state price $\overline{a}_1$ must be the same for all policies for the set under consideration, and by hypothesis p_{11}, p_{21} are the same under all policies as well. And by Lemma 2, $p_2/p_1, p_{12}/p_{11}$ and p_{22}/p_{21} are independent of policy as well. It then follows from (2.7) that the real price of durables can be changed by one of the policies under consideration if and only if the state price a_2 changes, and more specifically that p_3/p_1 increases if and only if a_2 increases as a result of the policy change. Moreover, $1 + r^{dur}$ must vary inversely with $(p_3 - p_2)/p_1$, so that r^{dur} falls if and only if a_2 increases.

Similarly, (2.6) implies that the quantity $(1+i)p_1$ can be changed if and only if a_2 changes, and more specifically that $(1 + i)p_1$ falls if and only if a_2 increases. It then follows from (2.16) that r similarly falls if and only if a_2 increases. Thus the expected real returns on both the risky durable and on riskless debt must fall if and only if a_2 increases. We can furthermore sign the difference between the percentage changes in the two expected returns, in the case of a given change in a_2. One observes that

$$\frac{1 + r^{dur}}{1 + r} = C \cdot \frac{a_1 \left(\frac{1}{p_{11}}\right) + a_2 \left(\frac{1}{p_{21}}\right)}{a_1 \left(\frac{p_{21}}{p_{11}}\right) + a_2 \left(\frac{p_{22}}{p_{21}}\right)},$$

where C is a positive constant (a function only of the prices $\{p_{sl}\}$ that are independent of policy). It follows from this that $(1 + r^{dur})/(1 + r)$ is an increasing function of a_2 (holding fixed a_1 and the $\{p_{sl}\}$). Hence the spread $\hat{r}^{dur} - \hat{r}$ increases if and only if a_2 increases. Since a_2 increases if and only if p_3/p_1 increases, the assertions in the first paragraph of the lemma have all been established.

In the case that there is no change in i, a decline in $(1 + i)p_1$ necessarily requires a decline (in the same proportion) in p_1 (and hence in p_2 as well, since p_2/p_1 is independent of policy). As shown above, an increase in a_2 necessarily implies a decrease in $(1 + i)p_1$ (and hence in p_1) by a factor that is larger than the factor by which $1 + r^{dur}$ declines (and hence by which $(p_3 - p_2)/p_1$ increases). It follows that the product

$$\frac{p_3 - p_2}{p_1} \cdot p_1$$

decreases if and only if a_2 increases. Hence $p_3 - p_2$ decreases, and since p_2 also decreases, it follows *a fortiori* that p_3 decreases, if and only if a_2 increases. Thus an asset-purchase policy that raises the nominal price of the durable in period 0 (whether one considers the pre-rental price p_3 or the post-rental price $p_3 - p_2$) must be one that lowers a_2, from which the conclusions stated in the second paragraph of the lemma then follow. □

A.11 Proof of Proposition 6

It follows from standard properties of offer curves that for all values $a_2 < a_2^{**}$, the points on the offer curve will involve $c_2^1 > k_2^1$, and that $\hat{c}_2^1(a_2)$ is a monotonically decreasing function over this range, increasing without bound as $a_2 \to 0$, Instead, for values $a_2 > a_2^{**}$, the function need not be monotonic, but necessarily all points on this part of the offer curve involve $c_2^1 < k_2^1$. Hence for any value $c_2^1 \geq k_2^1$, there is a unique $0 < a_2 < a_2^{**}$ such that $\hat{c}_2^1(a_2) = c_2^1$, and the required value of a_2 is monotonically decreasing as a function of c_2^1.

Moreover, assumption (3.7) implies that the indifference curve of household 2 through the endowment point A is steeper than that of household 1. Because the A-D equilibrium is unique, there can be only one point on the offer curve at which the slopes of the indifference curves are identical (namely, point E^*, corresponding to the A-D equilibrium), so for all values of a_2 in the interval $a_2^* < a_2 \leq a_2^{**}$, the slope of the indifference curve of household 2 is more negative than $-1/a_2$ at the point on the offer curve corresponding to a_2; and when $a_2 = a_2^*$, the slope is exactly $-1/a_2^*$.

Hence for any value of c_2^1 in the interval (3.8), there is a unique point on the offer curve, corresponding to a value of a_2 in the interval $a_2^* \leq a_2 \leq a_2^{**}$, for which $\hat{c}_2^1(a_2) = c_2^1$. This corresponds to an allocation in which household 1's expenditure plan is optimal, given the budget line defined by a_2; thus it will solve the problem for household 1 defined in condition (i) of Definition 3, as long as the lower bound defined by (3.4) is no higher than the assumed value of c_2^1. When $c_2^1 = c_2^{1*}$, household 2's expenditure plan is also optimal, given the budget line; thus it will solve the problem defined in condition (i) as well, as long as the lower bound defined by (3.4) for household 2 is no higher than the implied value $c_2^2 = \sum_h k_2^h - c_2^1$. If instead $c_2^1 < c_2^{1*}$, household 2 has an indifference curve through this point that is steeper than the budget line. This implies that household 2's plan is optimal among all those on the budget line that involve a value of c_2^2 no lower than $\sum_h k_2^h - c_2^1$. Thus household 2's plan solves the problem defined in condition (i) if and only if the lower bound defined by (3.4) for household 2 is exactly equal to $\sum_h k_2^h - c_2^1$.

This point on the offer curve, together with the associated value of a_2, accordingly constitutes an equilibrium neglecting short-sale constraints only if the lower bound for c_2^2 defined by (3.4) is exactly equal to $\sum_h k_2^h - c_2^1$, if $c_2^1 < c_2^{1*}$. This requires that

$$g_2^2 - \theta^2 \phi(\overline{a}_2) \omega e_3 = \sum_h k_2^h - c_2^1, \tag{A.4}$$

which requires that $\omega = \hat{\omega}(c_2^1)$, the value defined in (3.10). This is a feasible policy only if $0 \leq \hat{\omega}(c_2^1) < 1$, which is true if and only if the bounds (3.9) are satisfied. In the case that $c_2^1 = c_2^{1*}$, it is instead only necessary that the lower bound for c_2^2 be no higher than $\sum_h k_2^h - c_2^1$, which requirement is satisfied if and only if $\omega \geq \hat{\omega}(c_2^{1*})$. This defines a non-empty interval of feasible values for ω if the bounds (3.9) are satisfied (though actually only the upper bound in (3.9) is necessary in this case).

Thus any such point on the offer curve satisfies all of the conditions to be an equilibrium neglecting short-sale constraints, in the case of an asset-purchase policy of the kind defined in the proposition, as long as the lower bound for c_2^1 defined by (3.4) for household 1 is no higher than the assumed value of c_2^1. This requires that inequality (3.4) be satisfied by the proposed values of c_2^1, a_2, and ω. But the fact that (A.4) holds when $\omega = \hat{\omega}(c_2^1)$ implies that (3.4) holds as well (and is a strict inequality); this is just the observation already made earlier, that it is not possible for the leverage constraint (3.4) to simultaneously bind for both households. Moreover, the fact that the lower bound defined in (3.4) is a monotonically decreasing function of ω implies that (3.4) must also be satisfied in the case of any $\omega \geq \hat{\omega}(c_2^1)$. Hence all conditions for an equilibrium neglecting short-sale constraints are shown to be satisfied.

It has already been noted in the above derivation that the implied value of a_2 is a monotonically decreasing function of c_2^1. The fact that this then implies that the equilibrium value of p_3/p_1 will be a monotonically decreasing function of c_2^1 follows from the discussion in the proof of Proposition 5. □

A.12 Proof of Lemma 5

The offer curve of household 1 consists of the values (c_{01}^1, c_2^1) that satisfy the first-order condition

$$\frac{c_{01}^1}{c_2^1} = \left(\frac{\alpha_{01}}{\alpha_2}\bar{a}_2\right)^{1/\gamma} \tag{A.5}$$

and budget constraint 3.3) with equality, for any value of $\bar{a}_2$. Using (A.5) to substitute for c_{01}^1 in (3.3), and differentiating the resulting relationship between $\bar{a}_2$ and c_2^1 at any point where $c_2^1 > k_2^1$, one finds that

$$\begin{aligned} \eta^1_{a_2,c_2} &\equiv \frac{\partial \log \bar{a}_2}{\partial \log c_2^1} = -\frac{c_{01}^1 + \bar{a}_2 c_2^1}{\bar{a}_2(c_2^1 - k_2^1) + \gamma^{-1} c_{01}^1} \\ &> -\frac{c_2^1}{c_2^1 - k_2^1}. \end{aligned} \tag{A.6}$$

Here the inequality (A.6) relies upon the assumptions that $c_2^1 > k_2^1$ and $\gamma \leq 1$. Note that $\eta^1_{a_2,c_2} < 0$ as well.

Total differentiation of the relation (A.4) with respect to c_2^1 at any point $c_2^1 > k_2^1$ then yields

$$\frac{d\omega}{dc_2^1} = \frac{\Gamma}{\theta^2 \phi(\bar{a}_2) e_3}, \tag{A.7}$$

where

$$\Gamma \equiv 1 - \theta^2 \phi'(\bar{a}_2)[\bar{a}_2 \eta^1_{a_2,c_2}/c^1_2]\omega e_3$$
$$> 1 - \theta^2 \phi(\bar{a}_2)\omega e_3/(c^1_2 - k^1_2) = 1 - \frac{g^2_2 - c^2_2}{c^1_2 - k^1_2} = \frac{f^2_2}{c^1_2 - k^1_2} > 0. \qquad (A.8)$$

Here the inequality uses the fact that the definition (2.14) implies that

$$-\phi(\bar{a}_2) < \phi'(\bar{a}_2)\bar{a}_2 < 0,$$

and inequality (A.6); the next equality follows from the fact that (3.4) holds with equality for houssehold 2; and the final equality follows from the market-clearing relation (3.6). The final inequality then follows from the fact that $f^2_2 > 0$ and the assumption that $c^1_2 > k^1_2$. It then follows from (A.7) that ω is an increasing function of c^1_2. $\square$

A.13 Proof of Proposition 7

As explained in the proof of Proposition 6, equilibria neglecting short-sale constraints corresponding to values of c^1_2 in the interval (3.8) involve allocations on the offer curve of household 1, for budget lines corresponding to state prices in the interval $a^*_2 \leq \overline{a}_2 \leq a^{**}_2$; moreover, higher values of c^1_2 correspond to lower values of $\overline{a}_2$ (steeper budget lines). For any value of c^1_2 in the interval (3.8), the point on the offer curve is a point on the budget line above and to the left of the endowment point Ω. It then follows that a decrease in $\overline{a}_2$ (steepening the budget line through point Ω) rotates the budget line so that the point previously preferred by household 1 (indeed, all points on the previous budget line above and to the left of Ω) is now in the interior of household 1's budget set, so that a point that household 1 strictly prefers is now attainable. Hence the expected utility of household 1 must be monotonically increasing as one moves up the offer curve, so that $\hat{U}^1$ is a monotonically increasing function of c^1_2.

The function $\hat{U}^2(c^1_2)$ is obtained by evaluating the expected utility of household 2 as one moves up the offer curve of household 1. For values of c^1_2 close enough to k^1_2, the offer curve passes through the endowment point Ω with a slope of $-1/2a^{**}$, the slope of the indifference curve of household 1 through point Ω. The indifference curve of household 2 through point Ω is steeper, as noted earlier, as a consequence of (3.7). Hence near point Ω, the offer curve moves up and to the left from point Ω with a slope flatter than the indifference curve of household 2, so that the expected utility of household 2 is increasing as one moves up the offer curve. Hence $\hat{U}^2(c^1_2)$ must be an increasing function for values of c^1_2 close enough to k^1_2. On the other hand, the offer curve must approach the A-D allocation (point E^* in Figure 3) from below, from a direction that is to the left of the line $\overrightarrow{\Omega E^*}$, and therefore from the interior of the set of points that household 2 prefers to point E^* (a set bounded by the indifference curve of household 2 passing through E^*, which is tangent to the line $\overrightarrow{\Omega E^*}$). Hence the expected utility of household 2 is necessarily decreasing as one moves up the offer curve, at least from initial values close enough to the A-D allocation. Thus $\hat{U}^2(c_2^1)$ must be a decreasing function of c_2^1 for all values of c_2^1 close enough to c_2^{1*}. Finally, the total change in the value of $\hat{U}^2(c_2^1)$ as one moves up the offer curve from the endowment point to the A-D allocation must be positive, since the endowment point Ω is also a point on the budget line $\overrightarrow{\Omega E^*}$ associated with the A-D equilibrium, and household 2 must strictly prefer point E^* to this point, as shown in Figure 3. □

B Effects of Variation in Endowment Patterns: Numerical Examples

Here we present additional numerical illustrations of the way in which variation in endowment patterns affects the way in which households are constrained by the collateral constraints, and as a consequence, the way in which equilibrium is affected by central-bank purchases of the durable good. We begin by explaining why it suffices, in exploring the space of possible endowment patterns, to consider only the range of possible specifications of *endowment shares*.

B.1 Relevant Dimensions of Variation in Endowment Patterns: The Log Utility Case

An advantage of the log utility specification (4.1) is that in this case, the properties of the equilibria of interest do not depend on the aggregate endowments of the different goods at the different dates and in different states, but only upon the *shares* of the aggregate endowment of each type that are held by each of the household types. This reduces the number of parameters that need to be varied in order to explore all of the ways in which alternative endowment patterns can result in different types of equilibria.

Let us define endowment shares

$$s_1^h \equiv \frac{e_1^h}{\sum_h e_1^h}, \qquad s_3^h \equiv \frac{e_3^h}{\sum_h e_3^h}, \qquad s_{s1}^h \equiv \frac{e_{s1}^h}{\sum_h e_{s1}^h} \quad (s = 1, 2)$$

for each of the households h; feasibility requires that these each be non-negative, and that the sum of the shares of each type (over all households $h \in \mathcal{H}$) equal 1. Let us also define

$$s_d^h \equiv \frac{d^h}{p_{21} \sum_h e_{21}^h + p_{22} \sum_h e_3^h},$$

indicating the tax revenues that must be raised in period 1 to redeem the government debt endowment of household h, as a share of the value of the economy's aggregate endowment in state 2 (the state in which durables are less valuable). Then we can establish the following equivalence result.

Lemma 7 *Let $\mathcal{E}$ and $\mathcal{E}'$ be two economies, in each of which each household has preferences of the form (4.1). Suppose furthermore that the values of the share parameters $\{s_1^h, s_3^h, s_{s1}^h, s_d^h, \theta^h\}$ are the same for both economies, and that the price ratio $\rho \equiv p_{12}/p_{22}$ is also the same for both economies. (Note, however, that the aggregate endowments $\sum_h e_1^h, \sum_h e_3^h, \sum_h e_{s1}^h, \sum_h d^h$ and the future price-level commitments p_{s1} may be different in the two economies.) Then for any value of ω and any equilibrium of economy $\mathcal{E}$ associated with this policy, there is a corresponding equilibrium of economy $\mathcal{E}'$ for the same value of ω, in which the consumption shares*

$$\hat{x}_1^h \equiv \frac{x_1^h}{\sum_h e_1^h}, \quad \hat{x}_2^h \equiv \frac{x_2^h}{\sum_h e_3^h}, \quad \hat{x}_{s1}^h \equiv \frac{x_{s1}^h}{\sum_h e_{s1}^h}, \quad \hat{x}_{s2}^h \equiv \frac{x_{s2}^h}{\sum_h e_3^h}$$

are the same, the normalized intertemporal transfers[40]

$$\hat{y}_s^h \equiv \frac{\tilde{y}_s^h}{\sum_h k_s^h}$$

are the same, and the normalized state prices[41]

$$\hat{a}_s \equiv a_s \cdot \frac{\sum_h k_s^h}{\sum_h e^h}$$

are the same. It follows that the normalized real value of government debt in period 0,

$$\hat{d} \equiv \frac{d}{p_1 \sum_h e^h},$$

will be the same in the corresponding equilibria of the two economies, as will be the normalized real price of the durable asset,

$$\hat{p}_3 \equiv \frac{p_3}{p_1} \frac{\sum_h e_3^h}{\sum_h e^h}.$$

Hence conclusions about the effects of varying ω, both on the period 0 price level (and aggregate nominal expenditure) and on the equilibrium price (both nominal and real) of the durable asset, will be the same (in percentage terms) for both economies. Moreover, if the utility of household h in the equilibrium of economy $\mathcal{E}$ is u^h, then

[40] Here we again use the notation $\sum_h k_s^h \equiv \sum_h [f_s^h + g_s^h] = \sum_h [e_{s1}^h + (p_{s2}/p_{s1}) e_3^h]$.

[41] Here we again use the notation $e^h \equiv e_1^h + (p_2/p_1) e_3^h$ for the value of the household's "total non-durable endowment" in period 0.

the utility of that household in the equilibrium of economy $\mathcal{E}'$ is $u^h + \kappa^h$, where the constant κ^h depends only on the aggregate endowments of the two economies, but is the same for different equilibria corresponding to different asset-purchase policies ω. Hence utility comparisons between the equilibria associated with different asset-purchase policies are the same for both economies.

Proof. Preferences of the form (4.1) have the property that each household's utility $u^h(x^h)$ is equal to an expression of the form $\hat{u}^h(\hat{x}^h)$ plus a constant which depends only on the aggregate endowment pattern. Hence the household's decisions can be modeled as maximizing $\hat{u}^h$, and we can reformulate the household's decision problem in terms of its choice of a relative consumption plan $\hat{x}^h$, without having to specify the implied absolute consumption levels.

As above, the homotheticity of preferences implies that each household must choose to consume goods 1 and 3 in any state in the ratio of the aggregate endowments of those goods in that state, so that we can further reduce a household's choice of a relative consumption plan to its choice of an intertemporal relative expenditure plan $(\hat{c}^h, \hat{c}_1^h, \hat{c}_2^h)$, where we define

$$\hat{c}^h \equiv \frac{c^h}{\sum_h e^h}, \quad \hat{c}_s^h \equiv \frac{c_s^h}{\sum_h k_s^h}.$$

Log utility has the additional, stronger implication that

$$\frac{\sum_h e_1^h}{\sum_h e^h} = \frac{p_2}{p_1}\frac{\sum_h e_3^h}{\sum_h e^h} = \frac{\sum_h e_{s1}^h}{\sum_h k_s^h} = \frac{p_{s2}}{p_{s1}}\frac{\sum_h e_3^h}{\sum_h k_s^h} = \frac{1}{2} \tag{B.9}$$

in each state, as a consequence of (2.2).

The household decision problem can then be expressed as the choice of a plan $(\hat{c}^h, \hat{c}_1^h, \hat{c}_2^h, \hat{y}_1^h, \hat{y}_2^h)$ to maximize

$$\hat{u}^h = \log \hat{c}^h + \frac{1}{2}\log \hat{c}_1^h + \frac{1}{2}\log \hat{c}_2^h$$

subject to the constraints

$$\hat{c}^h + \hat{a}_1\hat{y}_1^h + \hat{a}_2\hat{y}_2^h \;\leq\; \hat{e}^h + \hat{a}_1\hat{f}_1^h + \hat{a}_2\hat{f}_2^h;$$

$$\hat{c}_s^h \;\leq\; \hat{g}_s^h + \hat{y}_s^h, \quad \text{for } s = 1, 2;$$

$$\hat{y}_2^h \leq \rho\hat{y}_1^h - \theta^h\frac{\rho - 1}{2}\omega;$$

and

$$\hat{y}_2^h \geq -\theta^h\hat{\phi}(\hat{a})\omega;$$

where

$$\hat{\phi}(\hat{a}) \equiv \frac{(\rho-1)\hat{a}_1}{2\hat{a}_1 + 2\rho\hat{a}_2},$$

and we define the additional normalized quantities

$$\hat{e}^h \equiv \frac{e^h}{\sum_h e^h} = \frac{s_1^h + s_3^h}{2},$$

$$\hat{f}_s^h \equiv \frac{f_s^h}{\sum_h k_s^h} = \frac{s_3^h}{2} + \rho^{s-2} s_d^h,$$

$$\hat{g}_s^h \equiv \frac{g_s^h}{\sum_h k_s^h} = \frac{s_{s1}^h}{2} - \rho^{s-2}\theta^h \sum_h s_d^h.$$

(Here we have repeatedly used (B.9) to simplify the expression of the constraints.)

An equilibrium can then be defined as a collection of normalized household plans and normalized state prices $\hat{a}_s$ such that each household's normalized plan solves the problem stated in the previous paragraph, and in addition, for each $s = 1, 2$,

$$\sum_h \hat{y}_s^h = \sum_h \hat{f}_s^h.$$

Since both the household problems and the market-clearing conditions can be written entirely in terms of the normalized household plans, the normalized state prices, the share parameters, the price ratio ρ, and the policy parameter ω, it follows that if economies $\mathcal{E}$ and $\mathcal{E}'$ have the same share parameters and the same value for ρ and ω, the possible equilibria must also be identical, to the extent that those equilibria are described in terms of the normalized household plans and the normalized state prices.

Moreover, (2.6) implies that

$$\hat{d} = [\rho^{-1}\hat{a}_1 + \hat{a}_2] \sum_h s_d^h,$$

so $\hat{d}$ will be the same in corresponding equilibria of the two economies as well. This implies that the percentage change in p_1 (and in aggregate nominal expenditure in period 0, the quantity Y defined in (1.7)) caused by a given change in ω will be the same for both economies. Similarly, (2.7) implies that

$$\hat{p}_3 = 1 + \left(\frac{p_3 - p_2}{p_1}\right) \frac{\sum_h e_3^h}{\sum_h e^h} = 1 + \frac{\hat{a}_1 + \hat{a}_2}{2},$$

so that $\hat{p}_3$ will be the same in corresponding equilibria of the two economies as well. This implies that the percentage change in both p_3 and in p_3/p_1 caused by a given change in ω will be the same for both economies.

Finally, each household's utility is given by the quantity $\hat{u}^h$ (which depends only on its normalized expenditure plan), plus a constant that depends only on the economy's aggregate endowment of the various goods in the various states. So the increase in $\hat{u}^h$ in moving from one equilibrium to another is equal to the increase in u^h. Thus our conclusions about the effects of asset-purchase policies on the welfare of each household type will also be the same for economies $\mathcal{E}$ and $\mathcal{E}'$. $\square$

Hence the alternative numerical values that need to be considered, if we assume preferences of the form (4.1) and only two household types, as in the examples considered in this section, can be reduced to eight real numbers: $\theta^1, s_1^1, s_3^1, s_{11}^1, s_{21}^1, s_d^1, s_d^2,$ and ρ.[42] If (as here) we restrict attention to economies in which the public debt is small,[43] we need only consider alternative points in a five-dimensional space.

In the examples below, we give particular attention to the consequences of variation in the values of s_{11}^1 and s_{21}^1, indicating the relative endowments of the non-durable good in each of the two possible states in period 1, holding fixed the household's period-0 endowments. Variation in these parameters allows us to show how the way in which the collateral constraints bind depends on the nature and degree of the heterogeneity in the hedging demands of the two household types, owing to differences in their state-contingent period-1 income unrelated to their portfolio choices.

In each of the figures, we consider how the character of equilibrium changes as s_{11}^1 varies between 0 and 1 (on the horizontal axis) and s_{21}^1 varies between 0 and 1 (on the vertical axis). Panel (a) of each figure shows how variations in the period-1 endowment pattern affect which collateral constraints bind, using the following shorthand to report the collateral constraints that bind in a given equilibrium. "SC^h" means that the short-sale constraint (2.12) binds for household h, while "LC^h means that the leverage constraint (2.13) binds for household h. Thus the notation "LC^1, SC^2" means that the leverage constraint of household 1 binds *and* that the short-sale constraint of household 2 binds, in the same equilibrium. We use the notation "AD" (since the equilibrium of our model coincides with the Arrow-Debreu equilibrium in this case) if neither constraint binds for any household.

Panel (b) each figure instead reports, for the same range of variation in the period-1 endowment patters, the signs of the derivatives with respect to ω of the expected utilities of each of the two household types, evaluated at the particular value of ω for which the figure is drawn. Plus and minus signs are used to indicate these signs: thus "+ +" means that the welfare of both types increases when ω is increased by a small enough amount (the case shown by a movement from Ω to E in Figure 3 above), "+ -" means that the welfare of household 1 increases while that of household 2 decreases

[42] Note that in the case of parameters indicating tax shares and endowment shares, a specification of household 1's share implies a value for household 2's share as well, as the shares must sum to 1.

[43] We assume a small positive value for d in our examples so that even when $\omega = 0$, it is possible to have as a positive supply of bank reserves M, as we assume throughout the paper.

(the case shown by a movement from E to E^* in Figure 3 in the text), and so on. In the case of an A-D equilibrium, to which Proposition 3 applies, we write "00" to indicate that both derivatives are zero.[44]

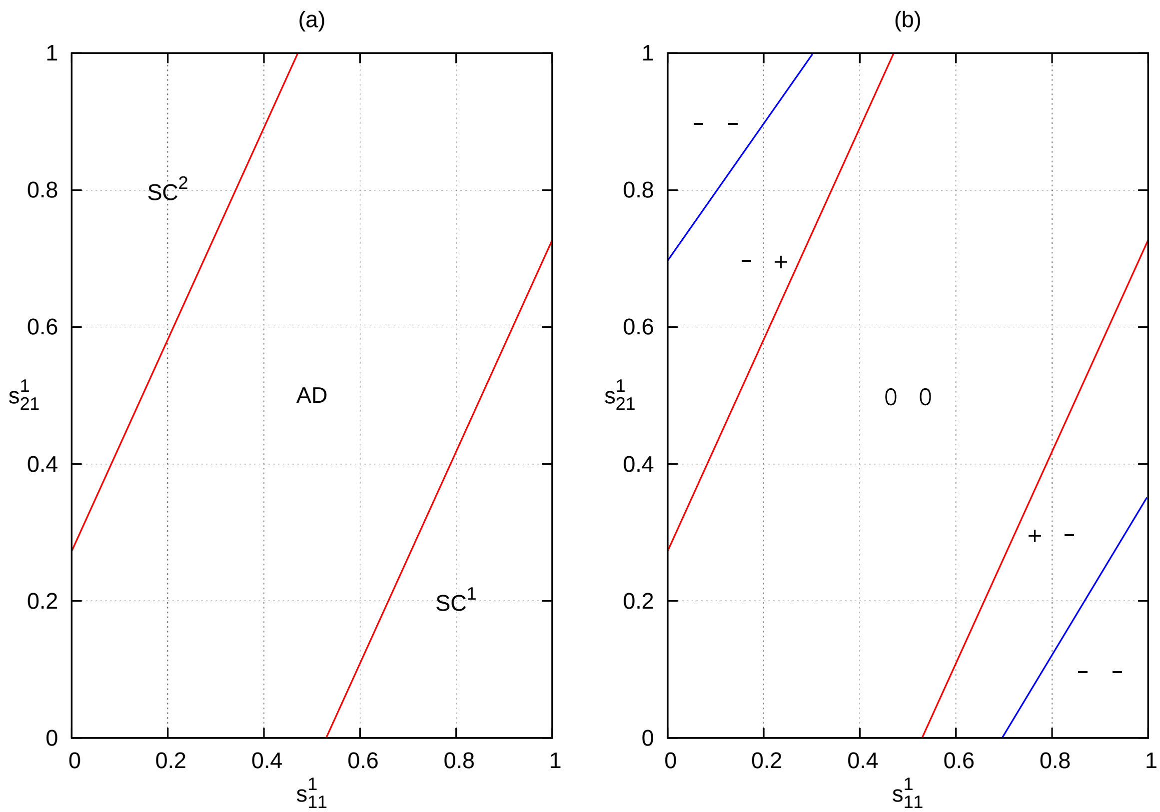

Figure 6: Example 1 with $\omega = 0$: (a) regions where collateral constraints bind; (b) welfare effects of a small increase in ω.

B.2 Example 1: Symmetric Initial-Period Endowments

In this example, we assume that both households have equal endowments of both the non-durable and durable goods in period 0 ($s_l^h = 0.5$, for $h = 1, 2$ and $l = 1, 3$), and that tax shares are equal as well ($\theta^h = 0.5$ for $h = 1, 2$). Endowments of government debt are also assumed to be equal, and of negligible magnitude.[45] We assume that

[44] There are never open regions of parameter space over which either derivative is exactly zero, except the region to which Proposition 3 applies, in which case both derivatives must be zero simultaneously.

[45] In the numerical results reported, we assume that $d^h/(1+i) = 0.0005$ for $h = 1, 2$; that $i = 0.1$; that $p_{11} = p_{21} = 1$; and that the aggregate non-durable endowment in state 2 is 6; so that $s_d^h = 0.000046$ for $h = 1, 2$.

the aggregate non-durable good endowment in state 1 is 15/7 times the aggregate endowment of the durable good, while in state 2 it is only 6/7 times the durable endowment; (B.9) then implies that $p_{12}/p_{11} = 15/7$ in state 1, while $p_{22}/p_{21} = 6/7$ in state 2. We also assume a period 1 monetary policy commitment to achieve the same inflation rate regardless of the state, so that $p_{11} = p_{21}$; hence $\rho = 5/2$ in this example.[46] The numerical example considered in Figure 4 in the text is a special case of the class considered in this section, corresponding to $s^1_{11} = 1, s^1_{21} = 0$ (the point in the lower right corner of the panels in Figures 6 through 8 below).

We first consider the kind of equilibrium that results in this case when $\omega = 0$ (the central bank holds none of the durable asset), for alternative assumptions about the households' relative shares of the period 1 non-durable endowment. Panel (a) of Figure 6 shows which collateral constraints bind in equilibrium, for alternative possible values of s^1_{11} and s^1_{21}.[47] As required by Proposition 4, in the symmetric case ($s^1_{s1} = 0.5$ for $s = 1, 2$), no collateral constraints bind, and we effectively have an A-D equilibrium. The figure shows that this continues to be true for specifications which are not perfectly symmetrical, but in which the endowment patterns of the two types are sufficiently similar. In particular, as long as the non-durable endowment shares are sufficiently *similar in the two states* that are possible in period 1, we have an A-D equilibrium, regardless of whether one household has a larger share of the period 1 endowment in both states.

The fact that the two households may have different motives to *save* (because one has more income in period 1 than in period 0, while the other has less) is not in itself a reason for any household's collateral constraint to bind. As long as each household's relative endowments in the two states is similar to the relative aggregate endowment in these states (that is, a non-durable endowment in state 2 that is about 40 percent of the size of the household's state 1 endowment), then households' desired intertemporal trade can largely occur simply by adjusting their holdings of the durable; and even if one household holds *all* of the period-1 non-durable endowment in both states (and therefore has the strongest possible motive to borrow), it can equalize its consumption share over time (consuming 5/8 of the aggregate supply of both goods in each state at each date) by selling half of its initial durable endowment in period 0, and thus entering period 1 (in either state) owning all of the non-durable endowment but only 1/4 of the aggregate supply of durables (worth 5/8 of the total supply of non-durable and durable goods, in either state). Thus for all points close enough to the diagonal in Figure 6(a), even the household with the smaller period-1 endowments continues to hold some of the durable and issues little debt, so that its

[46]Note that only the implied value of ρ matters for our conclusions below, and not our specific assumptions about aggregate endowments or monetary policy individually, as a consequence of Lemma 7.

[47]Here and in all of the numerical examples discussed below, there is a unique equilibrium for each endowment pattern and policy considered.

collateral constraint does not bind.

If, instead, the non-durable endowment shares are sufficiently different in the two possible states in period 1, one household's collateral constraint will bind, while the other remains unconstrained. The constrained household is the one that has a large share of the non-durable endowment in state 1, but a small share in state 2 (household 1 in the lower right region of the figure, household 2 in the upper left region); and the constraint that binds is the short-sale constraint (2.12). Thus in equilibria in the lower right region (labeled "SC^1"), household 1 is constrained in the way shown in Figure 2(b). Because household 1 has a larger endowment share in state 1, it would prefer a portfolio that paid off more in state 2 than in state 1; but this would require it to take a *short* position in the durable (that is worth more in state 1 than in state 2), which it cannot do because of the collateral constraint.

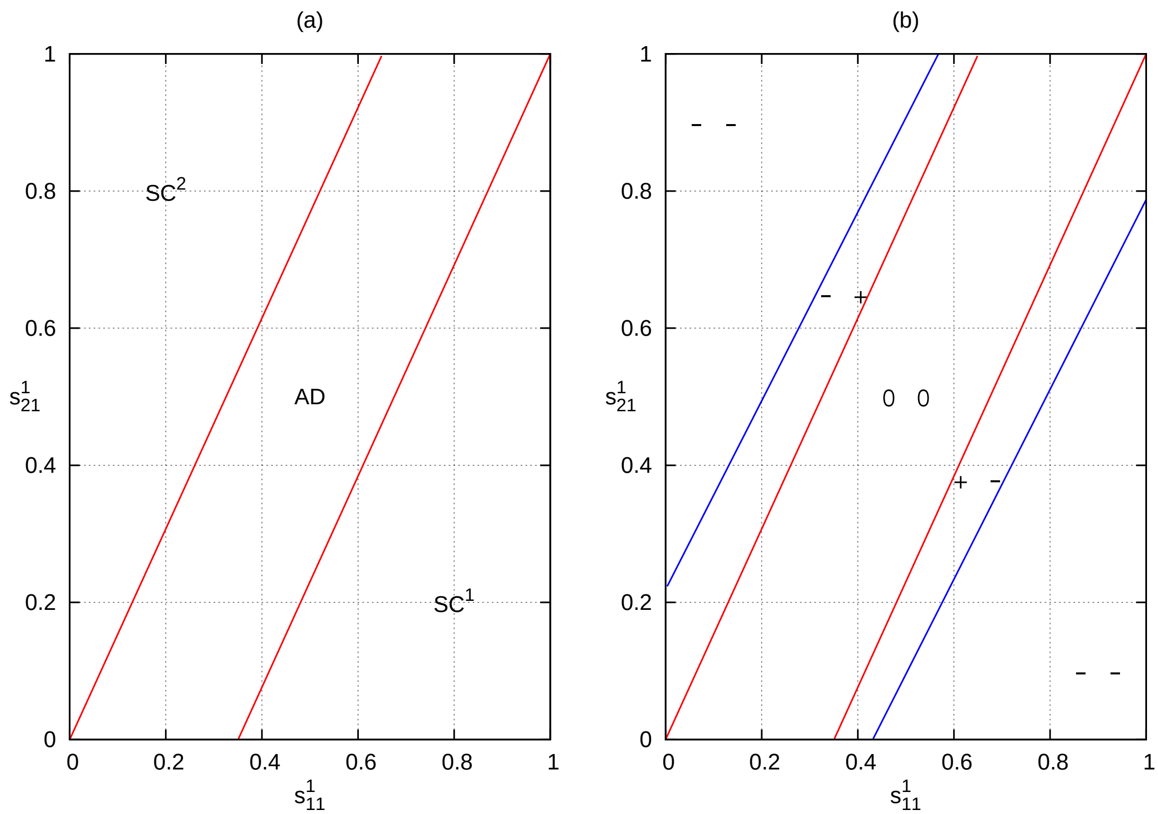

Figure 7: Example 1 with $\omega = 0.5$: (a) regions where collateral constraints bind; (b) welfare effects of a small increase in ω.

We turn to the question of how welfare is affected by small asset purchases by the central bank (a small increase in ω). Panel (b) of Figure 6 indicates for each of the cases the sign of the derivative of the utility level of each of the household types with respect to ω. In the case of economies in the diagonal region (labeled "AD") in panel (a), (sufficiently small) asset purchases have no effect on the equilibrium allocation of resources, by Proposition 3; hence there is no effect on welfare, and this region is labeled "00" in panel (b). When the relative endowments of the two types are sufficiently different in the two states, instead, asset purchases tighten the collateral constraint of the constrained household type, as shown in Figure 2(b).

The partial-equilibrium effect shown in that figure, however, does not suffice to sign the welfare effects. In order for markets to clear, the price of the durable rises, and this results in a positive income effect for the constrained household (a net seller of durables, since its short-sale constraint binds), and a negative one for the unconstrained household (that must be a net buyer). When the collateral constraint does not bind too tightly (so that the welfare effects of a small further tightening of the constraint are modest), this is the dominant effect, and the welfare of the constrained household is improved by central-bank purchases of the durable, while the welfare of the unconstrained household is reduced. Thus in Figure 6(b), the region just below and to the right of the diagonal region is labeled "+ -", indicating that the utility of household 1 increases while that of household 2 decreases.

In the case of an even more asymmetric endowment pattern, however, the distortion associated with the constrained household's binding collateral constraint is larger, and the consequences for welfare of further tightening of the constraint (shown in Figure 2(b) if one neglects the effects of price changes) are more substantial. For a sufficiently asymmetric endowment pattern, this becomes the dominant effect on the welfare of the constrained household; in such cases (indicated by the upper left corner and lower right corner of Figure 6(b)), the welfare of both household types is reduced by central-bank asset purchases. Such a policy change would thus be unambiguously undesirable.

In Figure 7, we instead assume an initial level of central-bank holdings of the durable of $\omega = 0.5$, and consider the effects of small additional asset purchases beyond that level. Figure 7 has the same format as Figure 6. In panel (a), we again observe that no collateral constraints bind for endowment patterns along the diagonal; but now the region labeled "AD" is a narrower strip around the diagonal. As the central bank purchases a larger share of the aggregate supply of the durable, the restrictions required in order for the collateral constraints not to bind become progressively more stringent; in fact (though we do not show this in a figure), for almost all possible endowment patterns, the collateral constraint eventually binds for one of the households, if ω is made large enough. Again, in this example, it is always the short-sale constraint rather than the leverage constraint that binds; and the welfare effects in the case of endowment patterns far enough from the diagonal are qualitatively the same as in the $\omega = 0$ case. However, when ω is larger, the degree of asymmetry in the period-1 non-durable endowments required in order for further asset purchases to reduce the welfare of both households is less extreme, as shown in

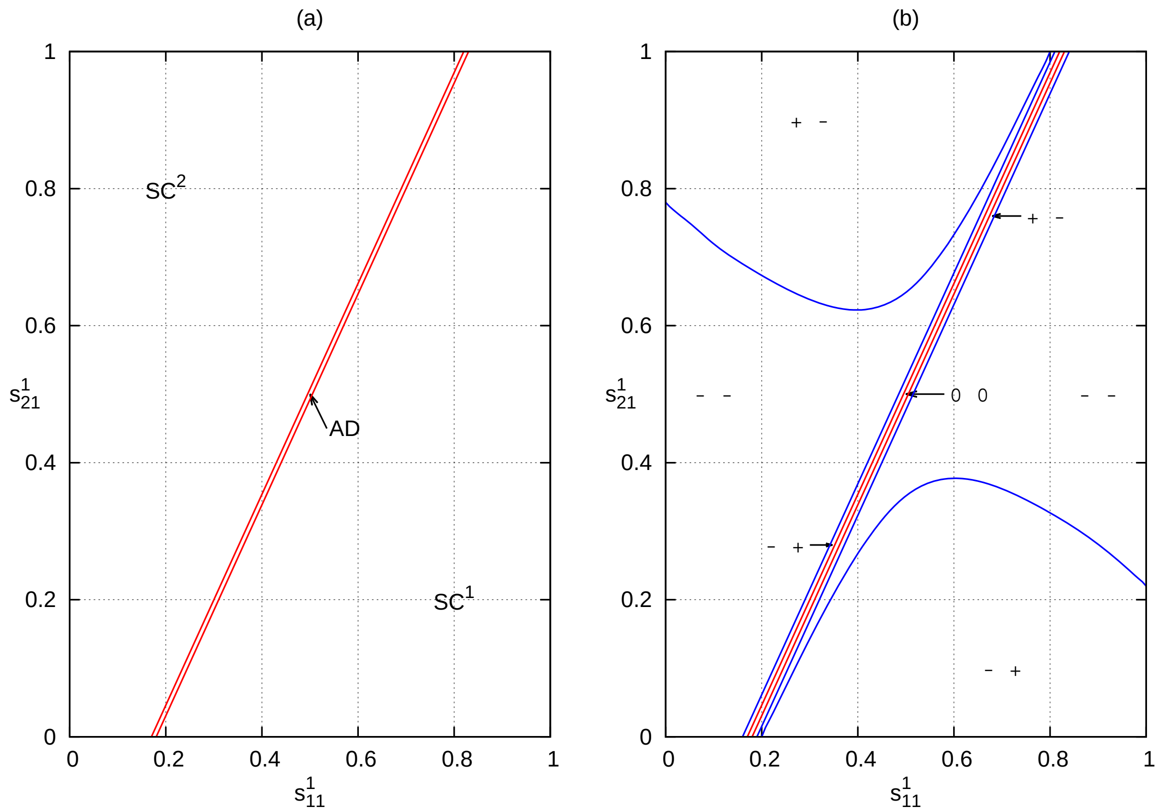

Figure 8: Example 1 with $\omega = 0.98$: (a) regions where collateral constraints bind; (b) welfare effects of a small increase in ω.

panel (b) of this figure.

Figure 8 shows how the results change if the central bank's share of the durable is increased still further. Further increases in ω continue to shrink the range of period-1 endowment patterns for which neither household's short-sale constraint binds; as shown in panel (a), by the time $\omega = 0.98$, both households' short-sale constraints fail to bind only in the case of endowments in a very narrow diagonal strip.[48] The regions near the "*AD*" region in which the short-sale constraint binds to such a mild extent that the household with the binding constraint *benefits* from additional asset purchases, despite the fact that such purchases tighten its short-sale constraint (e.g., the region below the diagonal region "00" in panel (b), labeled "+ -"), also become very narrow strips. In most of the plane, the welfare of the household with the binding short-sale constraint is reduced by further central-bank asset purchases.

[48] The "*AD*" region no longer includes the entire diagonal, but is instead a narrow strip somewhat steeper than the diagonal, because in our numerical example households do have positive (though small) initial endowments of money, and these are important for the location of the boundaries of the "*AD*" region for values of ω close enough to 1.

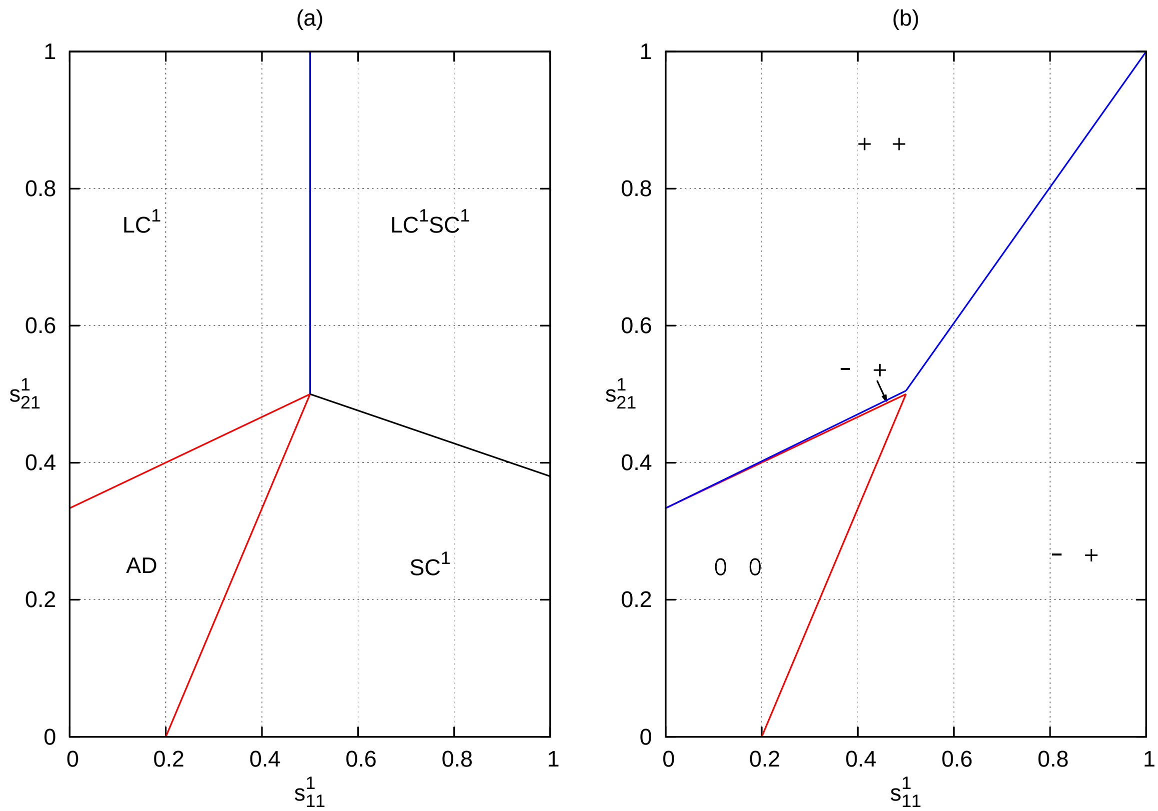

Figure 9: Example 2 with $\omega = 0$: (a) regions where collateral constraints bind; (b) welfare effects of a small increase in ω.

However, in the case of large enough values of ω, it is no longer always the case that the *unconstrained* household is harmed. As shown in panel (b) of this figure for the case $\omega = 0.98$, if the unconstrained household has a sufficiently large share of the aggregate endowment, then its welfare is increased by additional asset purchases, though the constrained household is harmed.

B.3 Example 2: Leverage-Constrained Investors

We now illustrate how a greater degree of asymmetry in the situations of the two household types can make possible equilibria in which the "natural buyers" of the risky asset are constrained (by the collateral requirement) in their ability to make as large a leveraged position in this asset as they would otherwise wish. Aggregate endowments (and hence the value of ρ) are the same in this class of numerical examples as in Example 1, and we again assume that $s_1^h = 0.5$ for $h = 1, 2$; but now we assume that only households of type 2 are initially endowed with the durable good ($s_3^1 = 0$). We also now assume asymmetric tax shares: $\theta^1 = 0.9, \theta^2 = 0.1$.

The government debt is also of the same (very small) size as in Example 1. But as in that example, we assume that initial endowments of government debt are distributed between the two types in proportion to their tax shares, so that now $d^1/(1+i) = 0.0009, d^2/(1+i) = 0.0001$. (The specific example considered in Figure 5 in the text is a special case of this class, corresponding to $s^1_{11} = 0, s^1_{21} = 0.5$. It thus corresponds to a point in the middle of the vertical axis on Figures 9 and 10.)

We again plot numerical results for alternative values of s^1_{11}, s^1_{21} in the plane. Figure 9 is for the case in which the central bank holds none of the durable asset. Along the diagonal in panel (a), we again have economies in which period-1 non-durable endowments are the same for the two households, and hence in which every household's endowment income in state 1 relative to that in state 2 is in the same ratio as the relative payoff of the durable asset in the two states. It thus again follows (in the limiting case of zero money endowments) that the A-D allocation could be supported purely through trade in the durable.

The difference is that now households of type 1 have no initial endowment of the durable, so that the required trade might involve a short sale of the durable by these households (which is not allowed by the collateral constraint). This is in fact the case if households of type 1 have a large enough share of the period-1 endowment income, so that they wish to borrow against their period-1 income in order to smooth their consumption level over time. Thus in Figure 9(a), the "*AD*" region no longer includes all of the diagonal. For points near the diagonal with $s^1_{11}, s^1_{21} < 1/2$, the A-D allocation can be supported with positive holdings of the durable by both types (and net positions near zero in the riskless asset for both); but for points near the diagonal with $s^1_{11}, s^1_{21} > 1/2$, we instead have an equilibrium in which *both* constraints (2.12)–(2.13) bind for type 1. This means that type 1 households choose to be at the corner of the grey region in Figure 1(a), corresponding to a zero position in both the durable and the riskless asset.

Figure 9(a) also differs from Figure 6(a) in that in the region above the "*AD*" region, we now have equilibria in which constraint (2.13) binds for households of type 1.[49] In these cases, households of type 1 have a substantial period-1 endowment in state 2, but not in state 1. This makes households of type 1 the "natural buyers" of the durable, as the durable (which is worth more in state 1 than in state 2) allows them to hedge their endowment risk, whereas the opposite is true for type 2 (who need to reduce their holdings of the durable in order to hedge their endowment risk).

In Example 1, this kind of asymmetry, if pronounced enough, resulted in an equilibrium in which the short-sale constraint bound for households of type 2. But now, with the durable asset initially held entirely by type 2, there is never a problem of household 2 wishing to take a short position in that asset. Instead, the constraint

[49] The existence of an "SC^1" region to the right of the "*AD*" region occurs for the same reason as in Example 1, and so requires no further discussion.

that prevents implementation of the A-D allocation is the *leverage* constraint of type 1: because households of type 1 initially own none of the durable (and do not have a large period-0 endowment of the non-durable good with which to purchase it, either), they need to borrow in order to acquire enough of the durable good for efficient risk-sharing with households of type 2. When the asymmetry of the period-1 endowments is severe enough, the required degree of leverage is no longer compatible with the collateral constraint. We thus obtain the possibility of an equilibrium in which the "natural buyers" of the risky asset are constrained in their ability to further leverage themselves in order to purchase as much of it as they would like. If in addition, as assumed here, θ^h is large for these investors, central-bank purchases of the durable will relax this leverage constraint to a significant extent.

As discussed in section 3, the observation that household 1's leverage constraint is relaxed does not suffice to determine the welfare effects of central-bank asset purchases. In the region where only household 1's leverage constraint binds, if the constraint does not bind too tightly (that is, at points near the boundary of the "AD" region), asset purchases reduce the welfare of household 1, while increasing the welfare of household 2, as in the passage from E to E^* in Figure 3 (but with the roles of the households reversed). Hence this region is labeled "-+" in Figure 9(b). For endowment patterns for which the constraint binds more tightly (points farther in the upper left corner of the figure), the welfare of both households is increased, as in the passage from Ω to E in Figure 3 (the region labeled "++" in Figure 9(b)). In this case, central-bank asset purchases are Pareto-improving.[50]

In the region where both the short-sale constraint and the leverage constraint bind for household 1 (that is, household 1 is at the corner of the set of feasible intertemporal transfers shown in Figure 1(a)), central-bank asset purchases relax the leverage constraint, but also *tighten* the household's short-sale constraint. Which of these effects is more important for the welfare of household 1 depends on which constraint binds more tightly. In the upper-left part of this region (the part closer to the region where only the leverage constraint binds), the most important effect is the relaxation of the leverage constraint, and a Pareto improvement results; but in the lower-right part of the region (the part closer to the region where only the short-sale constraint binds), the most important effect is the tightening of household 1's short-sale constraint, and the welfare of household 1 is reduced, though household 2 benefits from central-bank asset purchases.

Figure 10 shows how these figures change if instead we consider a situation in which the central bank holds half of the aggregate supply of the durable. The figures are qualitatively the same, but now the location of both the region in which the A-D allocation is achieved and the region in which a Pareto improvement occurs (the region "+ +" in panel (b) of the figure) are shifted up and to the left. The central

[50] The example shown in Figure 5 in the text belongs to this region.

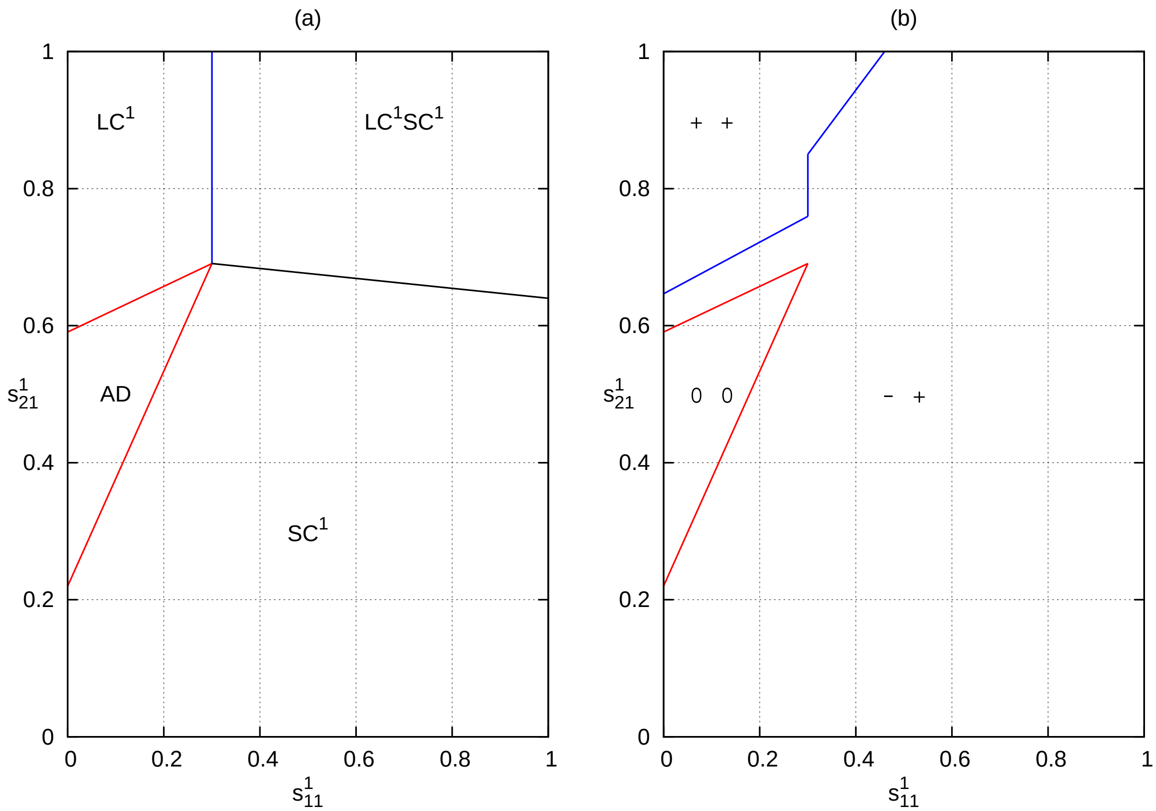

Figure 10: Example 2 with $\omega = 0.5$: (a) regions where collateral constraints bind; (b) welfare effects of a small increase in ω.

bank's policy increases the tax liability of household 1 in state 2 (the state in which the central bank suffers losses on the risky assets that it has acquired), while reducing it in state 1; this requires a more extreme asymmetry of the period-1 endowments in order for household 1 to be leverage-constrained, so that the region in which this occurs shifts up and to the left. Consequently, both the region in which asset-purchase policy is neutral and the region in which it is Pareto-improving are smaller parts of the plane in Figure 10(b) than in Figure 9(b).

If central-bank purchases are even larger, the picture changes even further, as illustrated by Figure 11 for the case $\omega = 0.98$. For large enough values of ω, it becomes possible for the short-sale constraint to bind for household 2 as well. In fact, for values of ω near enough to 1, the short-sale constraint binds for one household or the other, except in the case of fairly special endowment patterns (the two narrow slivers labeled "AD" and "LC^1" in Figure 11(a)). The conditions under which central-bank purchases of the durable are Pareto-improving become progressively more special as ω increases, and eventually this ceases to be possible for any endowment patterns of the kind considered in this example. For high enough values of ω, under any endowment

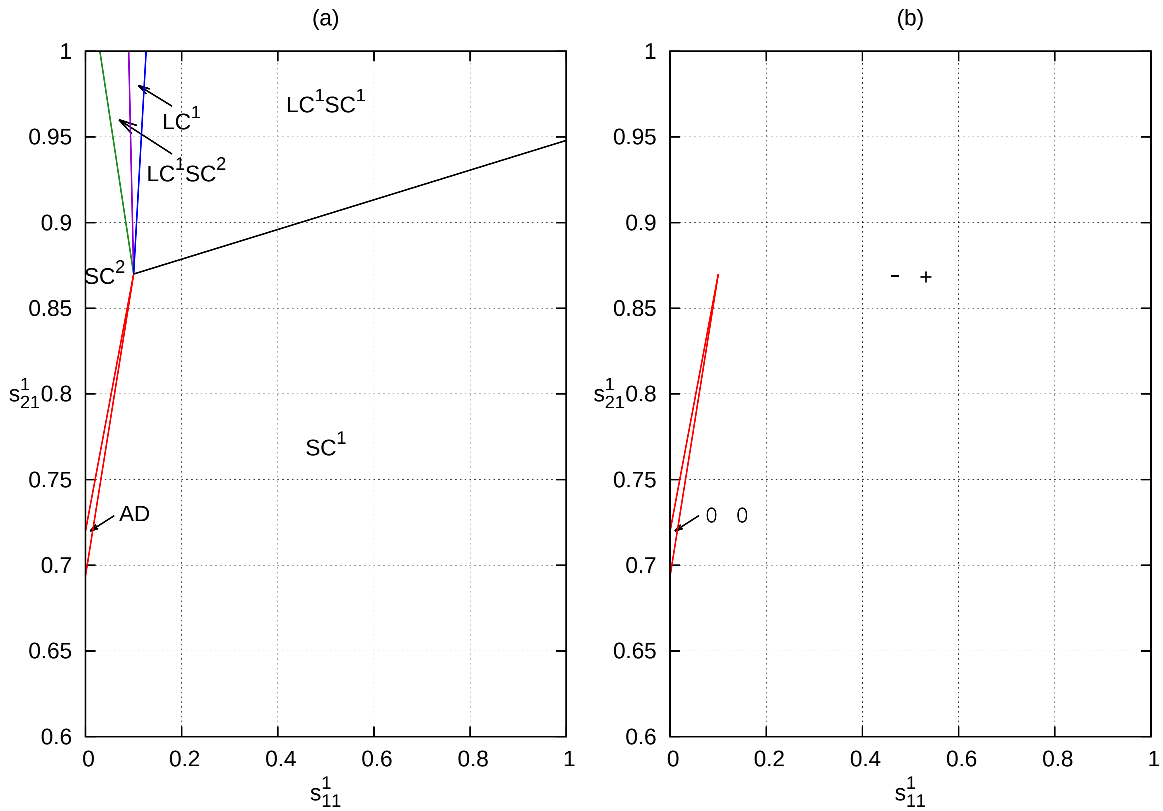

Figure 11: Example 2 with $\omega = 0.98$: (a) regions where collateral constraints bind; (b) welfare effects of a small increase in ω.

patterns other than the fairly special ones for which the A-D allocation continues to be achieved, further asset purchases always lower the welfare of households of type 1 (who largely bear the fiscal costs of the central bank's balance-sheet losses in state 2, and do not enjoy any income effect of increases in the market value of an initial endowment of durables), while increasing the welfare of households of type 2. This is illustrated for a particular endowment pattern in Figure 5 in the text.