

**Web Appendix to The Effects of Peer Group Heterogeneity on the Production of Human
Capital at West Point: A Structural Model of Peer Substitutability.**

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Section I. Roland Benabou (1996) presents an educational production model where peer influences are represented by a distribution of peer ability. His framework provides a tractable way of linking measures of dispersion in peer group distributions to the degree of substitutability between peer ability and educational outcomes. Based on the data and research design described in the main paper, I present a specific case of the Benabou model for educational attainment at West Point. Consider the following education production function for plebe-year math grades and plebe-year academic grade point averages (GPA):

$$(2) \quad g_i = q_i^\alpha \cdot \bar{q}_i^\omega \cdot \left[\frac{1}{N} \sum_{j=1}^N q_j^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma \cdot \beta}{\sigma-1}}$$

West Point assigns plebe (i) to a company with N other plebes (j). The N other plebes form the corresponding peer group (p). Plebes in a company study together and receive a math grade and a GPA at the end of the plebe-year, denoted by g_i . Each plebe has academic ability (q_i) obtained prior to entering West Point. Own ability (q_i), average ability of an individual's peer group (\bar{q}_i), and individual-level peer ability (q_j) for the distribution of peers are inputs in the production of grades with corresponding weights α , ω , and β . α is greater than zero, so that grades are increasing in own academic ability and $\frac{1}{\sigma}$ is a measure of the degree of substitutability between peers.

The sign on the cross partial derivative of the third term (in brackets) in equation (2) determines whether peer ability is a complement or substitute for own ability in this model. If the cross partial derivative is positive, then peer ability is a complement. If the cross partial

derivative is negative, then peer ability is a substitute. In this version of Benabou’s model, the sign on the measure of substitutability, $\frac{1}{\sigma}$, represents the sign on the cross partial derivative.¹¹ Therefore, peer ability is a complement when $\frac{1}{\sigma} > 0$ and a substitute when $\frac{1}{\sigma} < 0$.

The natural logarithm form of the model in equation (1) from the main paper is similar to what is found in the literature on peer effects.

$$(3) \quad Ln(g_i) = \alpha \cdot Ln(q_i) + \omega \cdot Ln(\bar{q}_i) + \frac{\sigma \cdot \beta}{\sigma - 1} \cdot Ln \left[\frac{1}{N} \sum_{j=1}^N q_j^{\frac{\sigma-1}{\sigma}} \right]$$

Most peer effect specifications regress an individual academic outcome, $Ln(g_i)$, on a measure of individual ability, $Ln(q_i)$, and a measure of average peer ability, $Ln(\bar{q}_i)$. In addition to this standard approach, the final term in equation (3) accounts for the distribution of peer ability. By expanding each q_j in equation (3) around the mean of plebe i ’s peer group \bar{q}_i , a second-order Taylor expansion results in the following linear model¹²:

$$(4) \quad Ln(g_i) = \alpha \cdot Ln(q_i) + (\beta + \omega) \cdot Ln(\bar{q}_i) - \frac{1}{2} \cdot \beta \cdot \frac{1}{\sigma} \cdot \left[\frac{Var(q_p)}{\bar{q}_i^2} \right] + R$$

The linear model in equation (4) captures the effect of own ability, average peer ability, and peer group heterogeneity as measured by the square of the coefficient of variation (variance of the peer group ability metric divided by the square of the mean). Estimating the sign of the peer group heterogeneity effect, $-\frac{1}{2}\beta\frac{1}{\sigma}$, and making some assumptions on the sign of β permit an estimate on the sign of the measure of substitutability, $\frac{1}{\sigma}$. β is one of two components of the average peer effect, $\beta + \omega$. Any effect of $\beta + \omega$ is likely positive because higher average ability peers are apt to be more beneficial to academic achievement than lower average ability peers.

¹¹ See Benabou (1996) page 590. Benabou defines $\frac{1}{\sigma}$ as the “cost of heterogeneity.”

¹² See Web Appendix Section II for details on the Taylor expansion. R represents higher order terms of the approximation and is assumed to be inconsequential.

Assuming a positive β , a negative sign on the estimate of $-\frac{1}{2} \beta \frac{1}{\sigma}$ implies $\frac{1}{\sigma}$ is positive or peer ability is a complement; a positive sign on the heterogeneity estimate implies $\frac{1}{\sigma}$ is negative or peer ability is a substitute.

In practice, estimating linear models in the form of equation (4) requires an educational environment where individuals have a known distribution of peers, random assignment, and robust pretreatment and contemporaneous measures of academic attainment to mitigate the selection, endogeneity, and common shock identification challenges. The educational environment at West Point as described in the main paper provides such a unique setting.

Estimating the model in equation (4) is akin to estimating a linear model as in equation (1) from the main paper.

$$(1) \quad Y_{ict} = \kappa + \theta_t + \lambda \cdot Z_{ict-1} + \delta \cdot \bar{Z}_{pt-1} + \gamma \cdot \hat{Z}_{pt-1} + \tau \cdot X_{ict-1} + \eta \cdot \bar{X}_{pt-1} + \varepsilon_{ict}$$

The estimate of δ , the average peer group effect, corresponds to $\beta + \omega$ and the estimate of γ , the heterogeneity effect, corresponds to $-\frac{1}{2} \cdot \beta \cdot \frac{1}{\sigma}$ from equation (3). Specifying the mean peer effect, δ with two components $\beta + \omega$ provides additional flexibility to this model. The inclusion of ω allows the average peer effect to equal zero without constraining the heterogeneity effect, $-\frac{1}{2} \cdot \beta \cdot \frac{1}{\sigma}$, to zero. In practice, it is possible that there is no effect of the mean, yet there is an effect of peer group heterogeneity. This is particularly important in this experiment because the random assignment process attempts to equalize peer group means. The benefit of relatively constant peer group means is that it allows for a clear comparison of peer group heterogeneity. The downside of this feature of the data is that it limits the ability of this experiment to identify average peer group effects. Therefore, for the purpose of interpretation, I make the assumption that β is the positive component of the mean effect and ω offsets β to

the true mean effect.¹³ To be clear, with this assumption, the model cannot estimate $\frac{1}{\sigma}$ precisely. However, it does allow for an estimation of the sign of $\frac{1}{\sigma}$, provided that any average peer effect is positive.

Appendix Table 1 contains estimates of the empirical model in equation (1) using the structural form of equation (4). In this case, Y_{ict} is the natural logarithm of the math grade or GPA, Z_{it-1} is the natural logarithm of the individual math SAT score, \bar{Z}_{pt-1} is the natural logarithm of the mean peer math SAT score, and \widehat{Z}_{pt-1} is the square of the coefficient of variation in peer group math SAT distributions.

Estimates of γ have a positive and significant effect of 3.97 and 1.29 log points for the math grade and the GPA, respectively. A structural interpretation of these estimates is that greater heterogeneity in plebe ability within a company, as measured by the square of the coefficient of variation for math SAT scores, improves academic scores. An increase of one standard deviation in the square of the coefficient of variation in peer math SAT scores positively affects the company average natural log of math grades by 10.8 percent and the company average natural log of GPA by 8.2 percent of a standard deviation.¹⁴ Even though the model does not provide enough information to estimate $\frac{1}{\sigma}$ precisely, it provides enough information to estimate the sign of $\frac{1}{\sigma}$. Assuming $\beta > 0$, the measure of substitutability, $\frac{1}{\sigma}$, is negative for both plebe math grades and GPA. A negative sign on $\frac{1}{\sigma}$ as defined in the Benabou model implies that peer ability is a substitute for own ability in the production of human capital at West Point.

¹³ In terms of the model this assumption is innocuous and still allows for the overall mean effect to be positive, negative, or zero. The primary impact of this assumption is on the interpretation of the results.

¹⁴ The company-level standard deviation is .11 for the natural log of math grades and .047 for the natural log of GPA.

Section II. Taylor Expansion Derivation of Equation (4)

The initial equation is: $g_i = q_i^\alpha \cdot \bar{q}_i^\omega \cdot \left[\frac{1}{N} \sum_{j=1}^N q_j^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma \cdot \beta}{\sigma-1}}$

Taking Logs: $\text{Ln}(g_i) = \alpha \cdot \text{Ln}(q_i) + \omega \cdot \text{Ln}(\bar{q}_i) + \frac{\sigma \cdot \beta}{\sigma-1} \cdot \text{Ln} \left[\frac{1}{N} \sum_{j=1}^N q_j^{\frac{\sigma-1}{\sigma}} \right]$

Expanding each q_j around \bar{q}_i , where $\bar{q}_i = \frac{1}{N} \sum_{j=1}^N q_j$:

$$\text{Ln}(g_i) = \underbrace{f(\bar{q}_i)}_{\text{(A)}} + \underbrace{\sum_{j=1}^N \frac{\partial f(\bar{q}_i)}{\partial q_j} \cdot (q_j - \bar{q}_i)}_{\text{(B)}} + \underbrace{\frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N \frac{\partial^2 f(\bar{q}_i)}{\partial q_j \partial q_k} \cdot (q_j - \bar{q}_i) \cdot (q_k - \bar{q}_i)}_{\text{(C)}} + R$$

Components of the Taylor Expansion:

(A) $f(\bar{q}_i) = \alpha \cdot \text{Ln}(q_i) + (\beta + \omega) \cdot \text{Ln}(\bar{q}_i)$

(B) This term goes to zero.

$$\frac{\partial f(\bar{q}_i)}{\partial q_j} = \beta \cdot \frac{1}{\bar{q}_i} \cdot \frac{1}{N} \Rightarrow \sum_{j=1}^N \frac{\partial f(\bar{q}_i)}{\partial q_j} \cdot (q_j - \bar{q}_i) = \beta \cdot \frac{1}{\bar{q}_i} \cdot \frac{1}{N} \sum_{j=1}^N q_j - \beta \cdot \frac{1}{\bar{q}_i} \cdot \bar{q}_i = 0$$

(C) $\frac{\partial^2 f(\bar{q}_i)}{\partial q_j \partial q_k} \Big|_{j=k} = \left(\frac{-\beta}{N \cdot \sigma \cdot \bar{q}_i^2} \right) + \left(\frac{-\beta \cdot (\sigma-1)}{N^2 \cdot \sigma \cdot \bar{q}_i^2} \right)$ $\frac{\partial^2 f(\bar{q}_i)}{\partial q_j \partial q_k} \Big|_{j \neq k} = \left(\frac{-\beta \cdot (\sigma-1)}{N^2 \cdot \sigma \cdot \bar{q}_i^2} \right)$

Rewriting and evaluating (C), where q_p represents the distribution of peers j :

$$\begin{aligned} &\Rightarrow \frac{1}{2} \sum_{j=1}^N \left(\frac{-\beta}{N \cdot \sigma \cdot \bar{q}_i^2} \right) (q_j - \bar{q}_i)(q_j - \bar{q}_i) + \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N \left(\frac{-\beta \cdot (\sigma-1)}{N^2 \cdot \sigma \cdot \bar{q}_i^2} \right) (q_j - \bar{q}_i)(q_k - \bar{q}_i) \\ &\Rightarrow \frac{1}{2} \left(\frac{-\beta}{\sigma \cdot \bar{q}_i^2} \right) \text{Var}(q_p) + 0 \quad \text{Second term goes to zero as in (2).} \end{aligned}$$

Final form of equation (4), where R denotes higher order terms:

$$\text{Ln}(g_i) = \alpha \cdot \text{Ln}(q_i) + (\beta + \omega) \cdot \text{Ln}(\bar{q}_i) - \frac{1}{2} \cdot \beta \cdot \frac{1}{\sigma} \cdot \left[\frac{\text{Var}(q_p)}{\bar{q}_i^2} \right] + R$$

**Web Appendix Table 1. Estimates of the Structural Model
Outcome Variable: Individual Plebe Math Grade or Plebe GPA**

	Math Grades		GPA	
	(1)	(2)	(1)	(2)
Peer Group Coefficient of Variation ² in Math SAT	3.971 (1.382)	4.405 (1.505)	1.287 (0.767)	1.458 (0.784)
Log of Peer Group Mean Math SAT	0.087 (0.270)	0.127 (0.284)	0.244 (0.141)	0.310 (0.151)
Log of Individual Math SAT	0.440 (0.058)	0.440 (0.058)	0.091 (0.030)	0.092 (0.030)
R ²	0.25	0.25	0.40	0.40
Observations	6,309		6,870	
Average Peer Group Scrambling Controls	No	Yes	No	Yes

Standard errors in parenthesis account for clustering at the company and year level. OLS estimates reflect regressions of the natural log of individual academic outcomes (as indicated in the column headings) on the natural log of own math SAT, the natural log of average peer math SAT, and the square of the coefficient of variation in peer math SAT scores. All specifications also include year dummies, a constant, and individual-level random scrambling controls: gender, race, recruited athlete, prep school, CEER, and WCS. Average peer group scrambling controls added as indicated. See Table 1 notes for sample description.